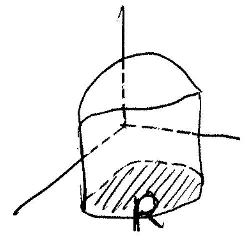
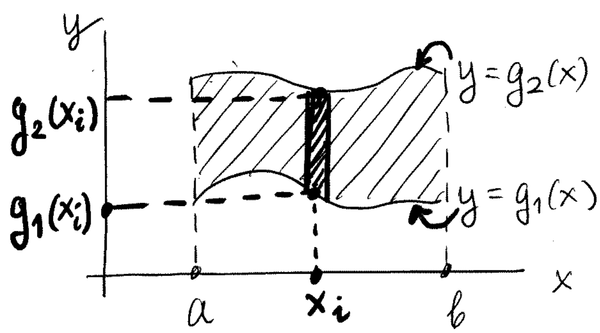
Sec. 15.2 Double integrals over non-rectangular regions

① Main result

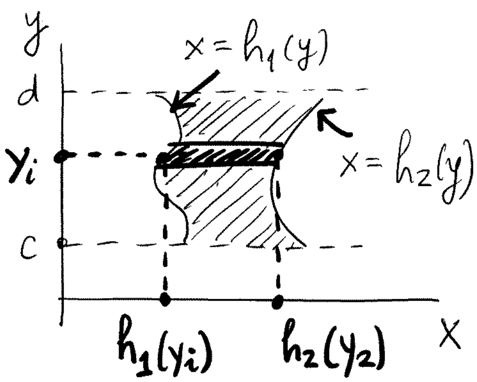
We will consider the situation when the integration region R in $\iint_R f(x,y)\,dA$ is of one of these Types:



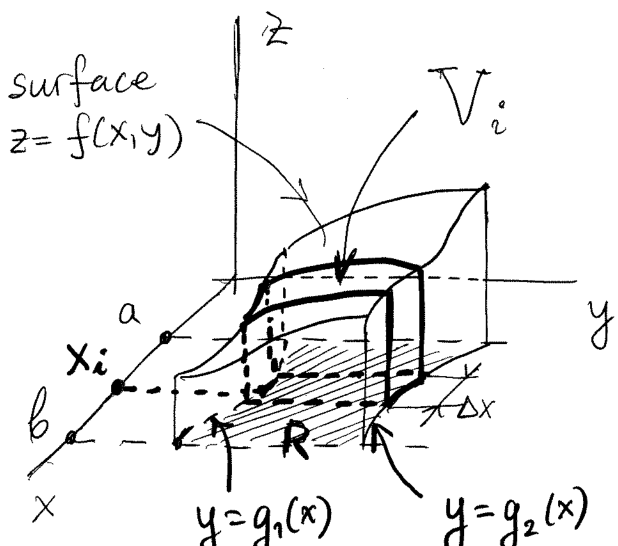
Type I



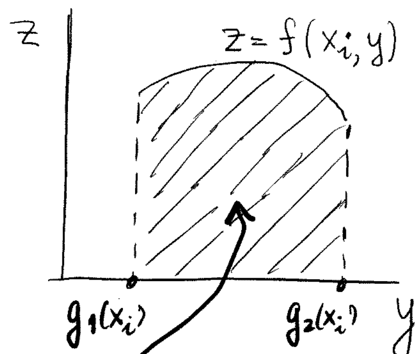
Type II



- Let us first consider how $\iint_R f\,dA$ can be written as an iterated integral when R is Type I:



View of the slice's face:



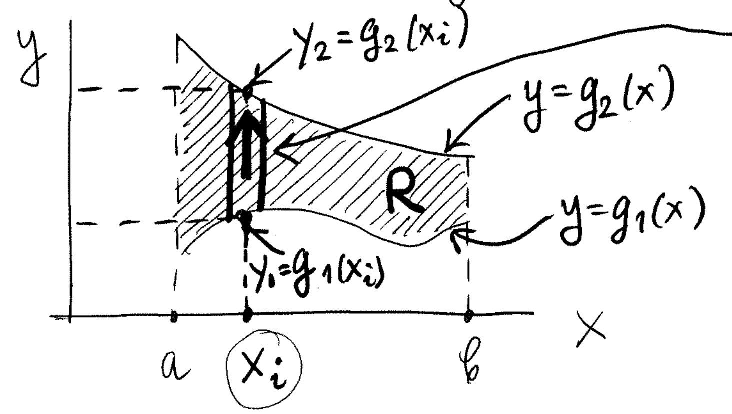
$$\text{Area} = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$$

$$\boxed{V = \sum_i V_i \approx \sum_i (\text{Area of face})_i \cdot \begin{pmatrix}\text{thickness}\\ \text{of slice}\end{pmatrix}}$$

Type I

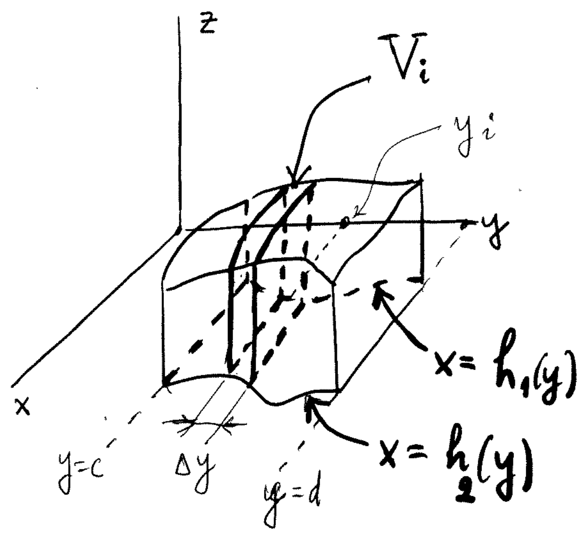
$$= \sum_i \underbrace{\left(\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy \right)}_{F(x_i)} \Delta x \xrightarrow[\Delta x \to 0]{} \boxed{\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx}$$

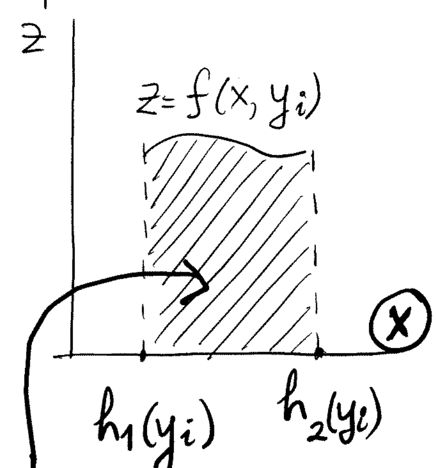
Thus, we first integrated $f(x,y)$ **along the vertical strip** and then added up the results for all such strips for $a \le x_i \le b$.



- We can similarly consider the case when R is Type II.

View of the slice's face:





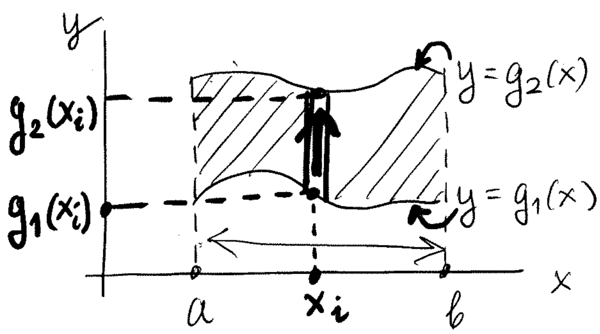
$$\text{Area} = \int_{h_1(y_i)}^{h_2(y_i)} f(x, y_i)\,dx$$

Then similarly to the calculation on p. 20-2:

$$V = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy \quad \text{(Type II)}$$

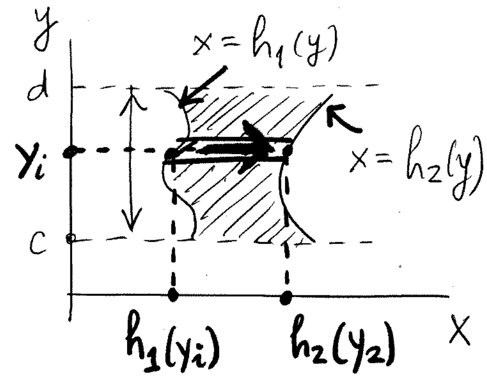
To summarize:

<u>Type I</u>



$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

<u>Type II</u>



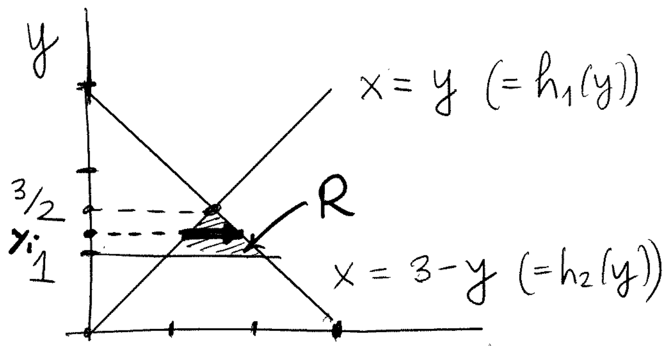
$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

<u>Mnemonic rules</u>:

- In the $\int \ldots dx$, the limits can<u>not</u> depend on x. They <u>can</u> depend on y or be constant.
- In the $\int \ldots dy$, the limits can<u>not</u> depend on y. They <u>can</u> depend on x or be constant.
- The limits of the <u>outer</u> $\int$ can NEVER depend on x or y. They can <u>only</u> be <u>constant</u>!

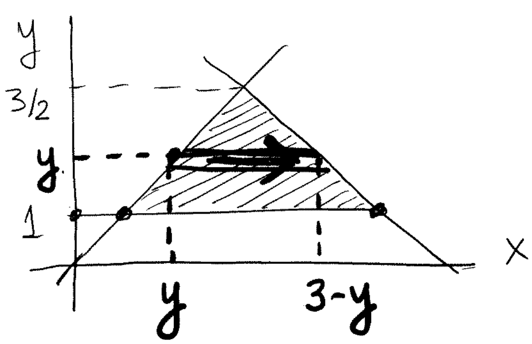
Ex. 1 Find $I = \int_1^{3/2} \left(\int_y^{3-y} (x+y)\,dx \right) dy$. ← Type II

Sol'n: 0) In Chap. 15, you will need to sketch in almost every problem. Here, one does not have to make it because the limits of integration are already determined. However, we will sketch just to practice.



Q: What is wrong about this sketch?

A: It is better to have a sketch that is not to scale but shows all details clearly then one that is to scale but crams details in a small area.



$$\text{Inner} \int = \int_y^{3-y} (x+y)\,dx = \underbrace{\int_y^{3-y} x\,dx}_{I_1} + \underbrace{\int_y^{3-y} y\,dx}_{I_2}.$$

$$I_1 = \frac{x^2}{2}\Big|_y^{3-y} = \frac{1}{2}\left((3-y)^2 - y^2\right) = \frac{9}{2} - 3y$$

$$I_2 = \int_y^{3-y} y\,dx = y \int_y^{3-y} 1 \cdot dx = y \cdot \left(x \Big|_y^{3-y} \right) = y(3-y-y) = 3y - 2y^2.$$

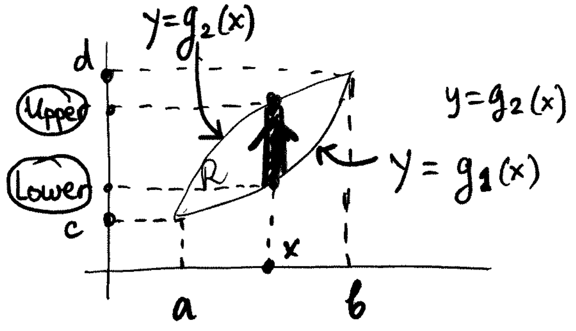
↑ const, when integrating over x

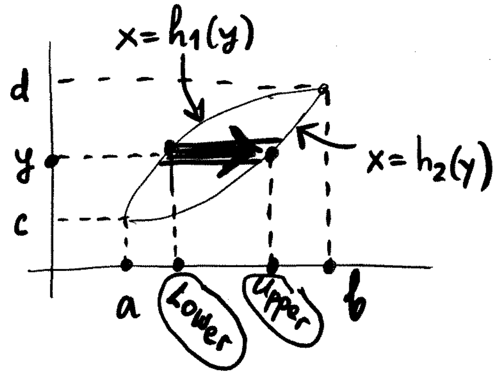
$$I = \int_1^{3/2} (I_1 + I_2)\,dy = \int_1^{3/2} \left(\frac{9}{2} - \cancel{3y} + \cancel{3y} - 2y^2 \right) dy$$

$$= \left(\frac{9}{2} y - \frac{2}{3} y^3 \right)\Big|_1^{3/2} = 17/12.$$

See also Ex. 1, 3 in book (similar).

Sometimes, region R is both Type I & Type II:





As Type I:

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

As Type II:

$$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

Then one chooses the integration order that is the more convenient (or, sometimes, the only one possible).

Ex. 2 Practice reversal of the integration order.

Express the given $\iint$ as an equivalent $\iint$ with the order of integration reversed.

$$I = \int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy$$

Sol'n:

0) SKETCH !

- The $\iint$ is set up as Type II. Comparing:

$$\int_0^2 \left(\int_{y^2}^4 f(x,y)\,dx \right) dy \equiv \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

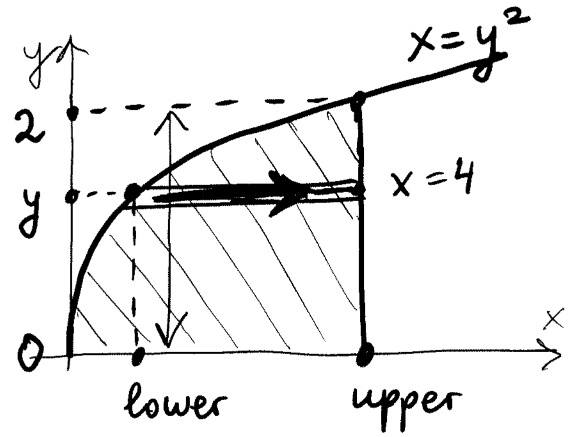
$\Rightarrow \quad h_1(y) = y^2 \equiv x_{lower}$

$h_2(y) = 4 \equiv x_{upper}$

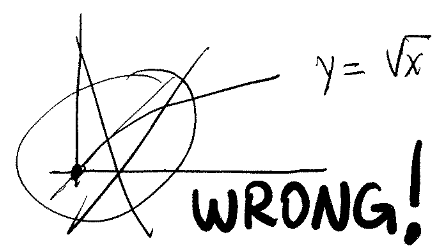
$c = 0 \equiv y_{min}$

$d = 2 \equiv y_{max}.$

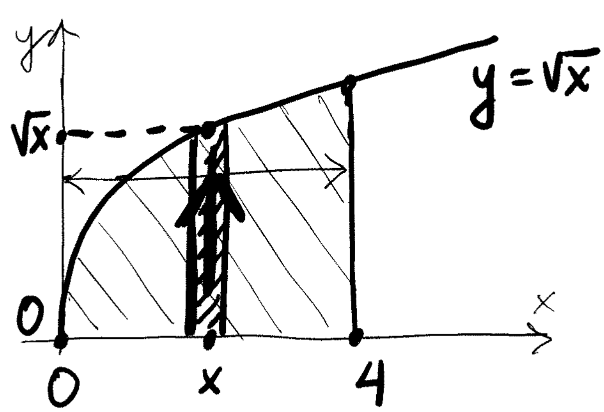
- The actual sketch, using h_1, h_2, c, d:



- **Note:**
$y = \sqrt{x}$ has the vertical tangent at $x = 0$.



1) Look at the sketch and re-cast the region as Type I.



Now:

$$\underset{y_{low}}{0} \le y \le \underset{y_{upp}}{\sqrt{x}}$$

$$\underset{x_{min}}{0} \le x \le \underset{x_{max}}{4}$$

Thus: $$I = \int_0^4 \left(\int_0^{\sqrt{x}} f(x,y)\, dy \right) dx.$$

! Never, ever try to manipulate "old" limits to obtain "new" ones.

Instead:

1. Make a sketch using the given limits.
2. Use the sketch (NOT the old limits!) to obtain new limits.

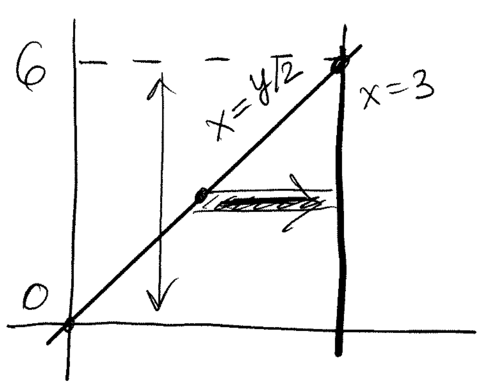
Ex. 3. Evaluate $I = \int_0^6 \left(\int_{y/2}^3 \sin(x^2)\, dx \right) dy$.

Sol'n: 1) Cannot do $\int \sin(x^2)\, dx$ — the anti-derivative of $\sin(x^2)$ is not an elementary function! So, switch the integration order.

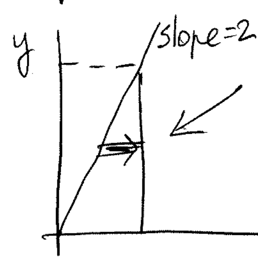
2) SKETCH!

- The region is set up as Type II.

$y/2 \le x \le 3$; $0 \le y \le 6$



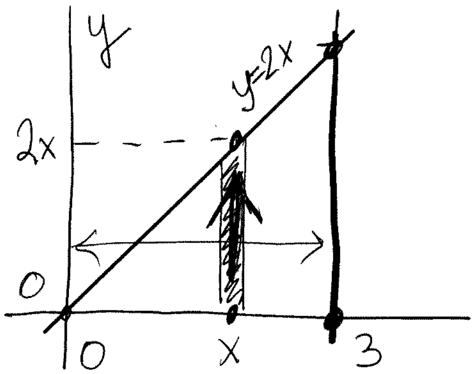
Recall the comment on p. 20-4: Sacrifice "to scale" for visibility of details:



Not terrible, but not as clear as the other one.

20-8

- Re-sketch and rewrite as Type I



$0 \le y \le 2x$ (y_{low}, y_{upp})

$0 \le x \le 3$ (x_{min}, x_{max})

$$I = \int_0^3 \left(\int_0^{2x} \sin(x^2)\, dy \right) dx.$$

3) Compute.

$$\text{Inner } \int = \int_0^{2x} \underbrace{\sin(x^2)}_{\text{= const when integrating over } y!} dy = \sin(x^2) \int_0^{2x} dy = \sin(x^2) \cdot 2x.$$

$$\text{Outer } \int = \int_0^3 \sin(x^2) \cdot 2x \cdot dx$$

1) $u = x^2$

$du = (x^2)' dx$

$= 2x\, dx$

$$= \int_0^9 \sin u \cdot du$$

limits for u, not x

$$= -\cos u \Big|_0^9$$

$$= -(\cos 9 - \cos 0)$$

$$= 1 - \cos 9$$

2) Change limits!

$u_{low} = 0^2 = 0$

$u_{upp} = 3^2 = 9$

Answer: $\iint = 1 - \cos 9.$ ✓

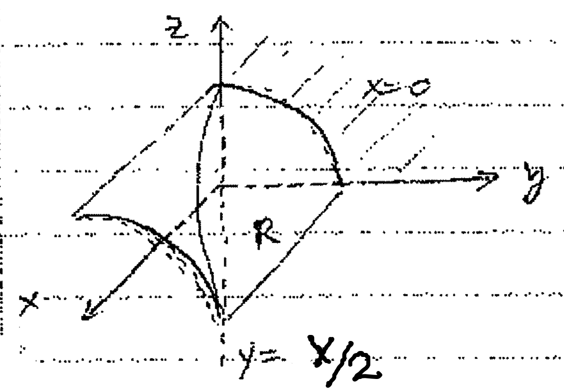
See also Ex. 2, 5 in book.

When finding the volume of a solid, one may need to <u>determine</u> the int. limits. See the next example.

Ex. 4 Find the volume of a solid bounded by the cylinder $y^2+z^2=4$ and the planes $y = x/2$, $x=0$, $z=0$ in the first octant.

Sol'n:

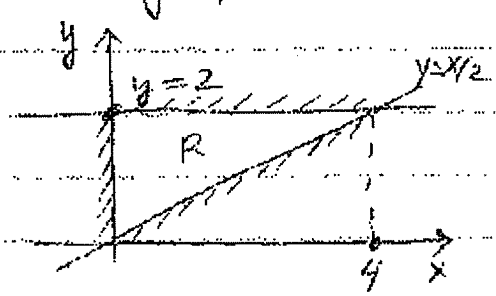
1) Sketch the solid



- $y^2+z^2=4$ – circular cylinder of radius 2, axis = x-axis
- $z=0$
- $x=0$
- $y = x/2$

$Z = + \sqrt{4-y^2}$ ← the part above the xy-plane

2) Find the boundaries of region R in the xy-plane



(a) If the top surface is $z=f(x,y)$, set $z = f(x,y) = 0$ ← bottom (in this problem)
(intersection w/ xy-plane)

$\sqrt{4-y^2} = 0 \Rightarrow$

$y^2=4 \Rightarrow y = +2$ or $\cancel{-2}$ ← need 1st octant only

(b) the other boundaries are given: $y = x/2$; $x=0$

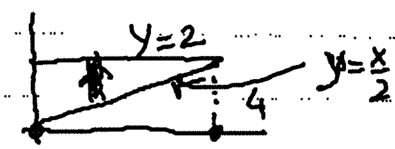
- This is both Type I and Type II region, so may need to try both.

3) $V = \iint_R \left(z_{top}(x,y) - z_{bot}(x,y) \right) dA$

$z_{top}(x,y) = \sqrt{4 - y^2}$, $\quad z_{bot}(x,y) = 0$

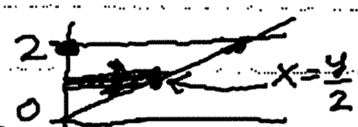
$= \iint_R \sqrt{4-y^2}\, dA.$

(a) Try Type I setup:



$V = \int_{x=0}^{x=4} \left(\int_{y=x/2}^{y=2} \sqrt{4-y^2}\, dy \right) dx$. Doable, but requires trig. sub (Calc. II).

So try Type II-setup:



(b) $V = \int_{y=0}^{y=2} \left(\int_0^{2y} \sqrt{4-y^2}\, dx \right) dy$

const. when integrating over x

$= \int_0^2 \sqrt{4-y^2} \cdot \left(x \Big|_0^{2y} \right) dy = \int_0^2 \sqrt{4-y^2} \cdot \underbrace{2y\,dy}_{-du}$

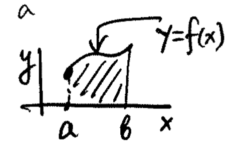
$= \int_4^0 \sqrt{u}\,(-du) = \int_0^4 u^{1/2} du = \frac{u^{3/2}}{3/2} \Big|_0^4 = \frac{16}{3}.$

$u = 4 - y^2$
$du = (4-y^2)' dy$
$= -2y\,dy$
$u_{low} = 4 - 0^2 = 4$
$u_{upp} = 4 - 2^2 = 0$

② Properties of double integrals (especially $\boxed{6}$, $\boxed{11}$)
Read the last 1.5 page of Sec. 15.2, including Ex. 6.
— they are straightforward and same as for the single integral.

(3) Double integrals using symmetry

Idea: The inner $\int$ in $\iint_R$ can be viewed as **area** under some curve ($\int_a^b f(x)dx$), and this sometimes can be used to find it w/o a calculation.

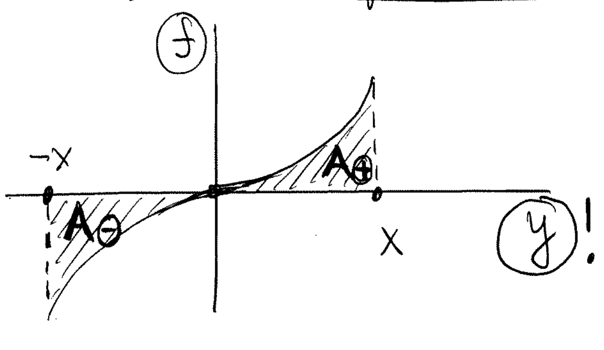


(This was considered in Calc. I: end of Sec. 5.5.)

Ex. 5 Find $\int_0^{\pi} \left(\int_{-x}^{x} \underbrace{y^3 \sin(xy^2)}_{f(x,y)} dy \right) dx$.

Note that $f(x,y)$ is the same as in Ex. 1 of Sec. 15.1-B (but the limits of $\iint$ were different). There, we had to switch the integration order ($dy\,dx \to dx\,dy$) so as to avoid difficult integration. Here, we will use the special form of the inner limits to compute the integral w/o a calculation.

Sol'n: 1) Sketch the function $f(x,y)$ within the inner integration limits.



$f(x,y) = y^3 \cdot \sin(xy^2)$

keep as const

This function of y is odd: $f(x,-y) = f(x,y)$.

Indeed:

$$f(x,-y) = \underbrace{(-y)^3}\cdot \sin(x\underbrace{(-y)^2})$$

$$= -y^3 \cdot \sin(xy^2) = -f(x,y). \checkmark$$

Therefore, in the figure above:

Area under f from $y=0$ to $y=x$

=

Area above f from $y=-x$ to $y=0$

=

$-$(Area "under" f from $y=-x$ to $y=0$).

Thus, total area "under" f from $-x$ to x

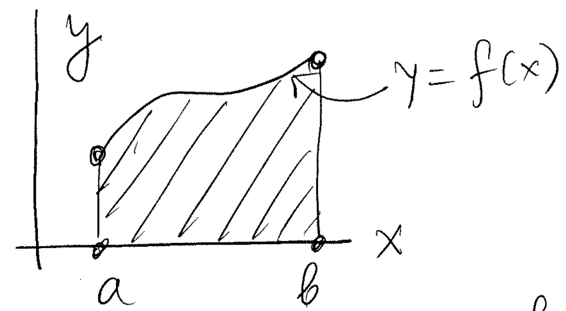
$= 0$! (The "!" expresses the excitement about this answer; it does not mean "zero factorial". ☺)

- Note: In some HW problems, you'll find a similar situation.
  But in others, you will be asked to recognize the area as that of a familiar shape and thus find it w/o a calculation.

4) Double integral as area

In Sec. 15.1A we introduced a $\iint_R$ as some volume. Now we will show that <u>some</u> $\iint_R$'s can also be viewed as <u>area</u>.

Analogy from Calc. I



We know that

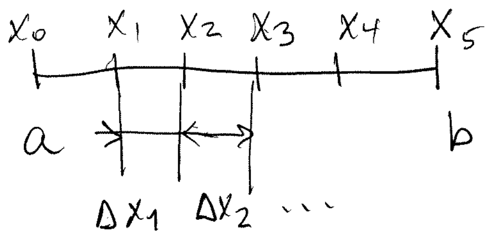
$$\int_a^b f(x)\,dx = \text{Area under } y=f(x) \text{ above } [a,b].$$

But what about $\int_a^b dx$?

By direct calculation: $\int_a^b 1\cdot dx = x\Big|_a^b = (b-a)$

$= \text{length of } [a,b].$

By original definition:

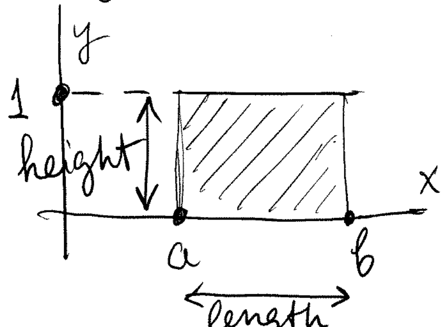


$$\int_a^b dx = \lim_{\Delta x \to 0} \sum_i \Delta x_i$$

$= (\Delta x_0 + \Delta x_1 + \dots \Delta x_n)$

$= \text{length of } [a,b].$

"By area":



$$\int_a^b 1\cdot dx = \text{Area under } y=1 \text{ over } [a,b]$$

$= \text{length} \cdot \cancel{\text{height}}^{\,1}$

$= \text{length (of } [a,b]).$

Thus, we will generalize:

Calc. I

$\int_a^b f(x)\,dx$ $\downarrow$ (Area under $y=f(x)$ above $[a,b]$)

$\int_a^b 1\cdot dx$ $\downarrow$ Length of $[a,b]$.

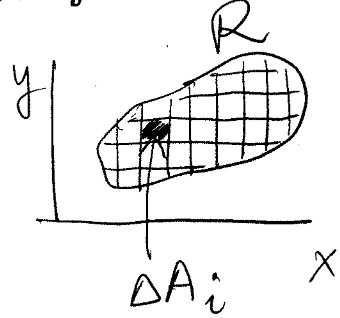
Calc. III

$\iint_R f(x,y)\,dA$ $\downarrow$ Volume under $z=f(x,y)$ above R

$$\boxed{\iint_R 1\cdot dA \downarrow \text{Area of } R.}$$

- Interpretation by definition:

$$\iint_R dA = \lim_{\Delta A\to 0} \sum_i \Delta A_i$$
$$= \text{Sum of areas } \Delta A_i$$
$$= \text{Total area of } R.$$



- Interpretation "by volume":

$$\iint_R 1\cdot dA = \text{Volume under } z=1 \text{ above } R$$
$$= \begin{pmatrix}\text{Area of}\\ \text{base}\end{pmatrix}\cdot \cancel{\text{height}}^{\,1}$$
$$= \text{Area of } R.$$