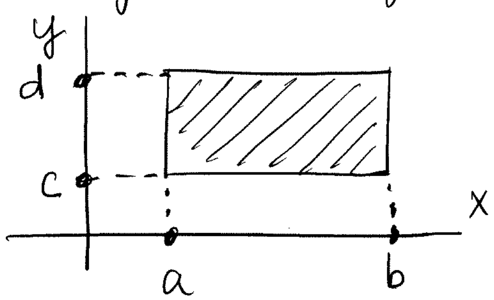
Sec. 15.3. Double integrals in polar coordinates.

① Main result

a Motivation

In cartesian coord's, the most convenient integration region is a rectangle:

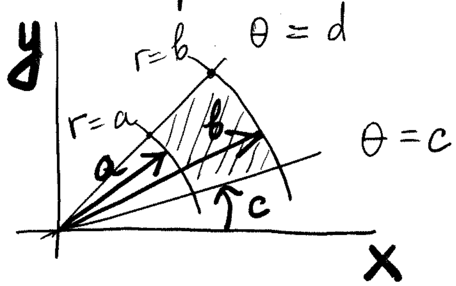


Bounds:

$x_{min} = a$, $x_{max} = b$

$y_{min} = c$, $y_{max} = d$

In polar coord's, the most convenient region is a shape that is a sector in cartes. coord's:

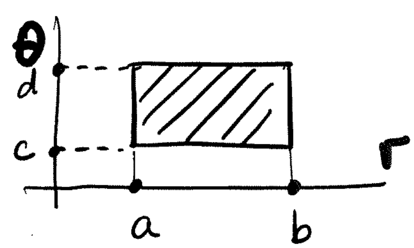


But this shape in polar coord's is a rectangle!

$r_{min} = a$, $r_{max} = b$

$\theta_{min} = c$, $\theta_{max} = d$

Same



This is why one tends to use polar coord's when the int. region has elements of circular symmetry.

6 Formula

Q: How can we express $\iint_R f(x,y)\,dA$ in polar coord's?

A (partial): • We know $f(x,y) = f(r\cos\theta, r\sin\theta)$;

• We can describe R via its boundaries in polar coord's.

• But what is dA in terms of (r,θ)?

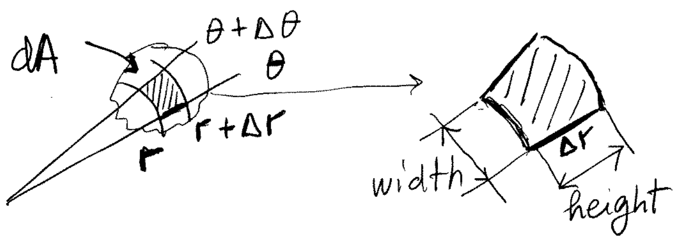
Note: $dA \neq dr \cdot d\theta$!!! Why not?

Because of dimension: units of dA = length^2 (area!);

units of $dr \cdot d\theta$ = length $\cdot$ 1 (θ is measured in radians, which is considered nondimensional)

= length, $\neq$ length^2.

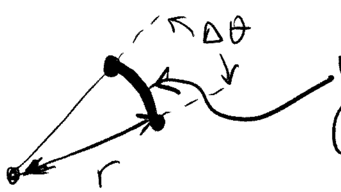
• Derivation of dA



When $dr, d\theta$ are small, dA is approximately a rectangle,

$\Rightarrow dA \approx \text{width} \cdot \text{height} = (r \cdot \Delta\theta) \cdot \Delta r$

$= r \cdot \Delta r \cdot \Delta\theta$



width $= r \cdot \Delta\theta$ (Sec. 13.3)

In the limit $dA \to 0$:

$$\boxed{(dA)_{polar} = r\,dr \cdot d\theta}$$

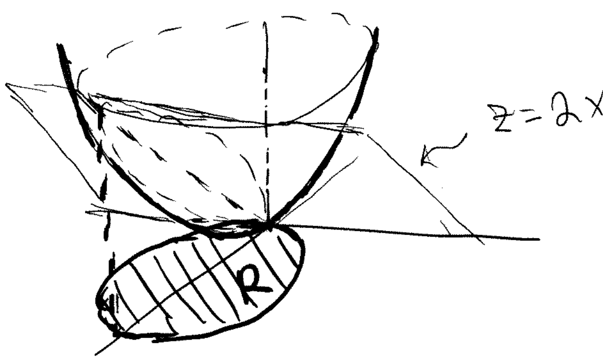
$$\boxed{\iint_R f(x,y)\,(dx\,dy) = \iint_R f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta}$$

MUST MEMORIZE!

You'll need to review polar eqs. of two types of circles in polar coords:
(i) centered at (0,0); (b) passing through (0,0).
You'll also need the equation of a ray from (0,0).
MUST SEE Sec. 10.3, Examples 4, 5, 6 + Problems 17, 25, 33.

Ex. 1 Find the volume of a solid bounded by the plane $z=2x$ and paraboloid $z=x^2+y^2$.

Sol'n: 0) Sketch



1) General formula:

$$V=\iint_R \left(z_{top}-z_{bottom}\right)dA$$

From sketch:

$$z_{top}=2x, \quad z_{bottom}=x^2+y^2.$$

2) Find region R.

The boundary is the projection on xy-plane of the intersection line of the top & bottom surfaces:

$$\mathbf{z_{top}=z_{bottom} \Longrightarrow \text{boundary of } R.}$$

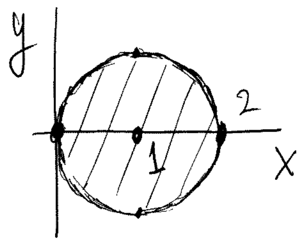
(i) Do it in Cartesian coord's. This will allow you to tell the shape of R as some familiar curve.

(ii) Then transform the boundary into polar coord's to do the integration.

So: (i) $2x = x^2 + y^2$

Complete the square (high school!)

$$x^2 - 2x + y^2 = 0$$
$$\underbrace{x^2 - 2x + 1} + y^2 = 0 + 1$$
$$\boxed{(x-1)^2 + y^2 = 1}$$



Thus, R = circle, center @ $(1,0)$, radius $= 1$.

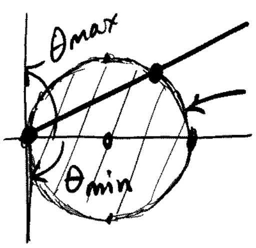
(ii) Same in polar:

$$x^2 + y^2 = 2x \Rightarrow r^2 \underbrace{\cos^2\theta + r^2 \sin^2\theta} = 2(r\cos\theta)$$
$$r^2 \cdot 1 = 2 \cdot r \cdot \cos\theta \Rightarrow$$

(cancel by r, since we are not interested at point $r=0$)

$$\boxed{r = 2\cos\theta}$$

3) Set up integration limits; use the sketch from 2).



In polar coords, the <u>inner limits</u> will always be <u>for **r**</u>.

@ $\theta = $ const (along the ray):

$r_{min} = 0$, $r_{max} = 2\cos\theta$ (eq. of the curve)

$\theta_{min} = -\pi/2$, $\theta_{max} = \pi/2$.

4) Set up the int'l in polar coord's:

$$V = \iint_R (z_{top} - z_{bot})\, dA = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos\theta} (\underbrace{2r\cos\theta}_{2x} - \underbrace{r^2}_{x^2+y^2}) \cdot \underbrace{r\,dr\,d\theta}_{dA} \right)$$

θ_{max}, r_{max} (at <u>some</u> θ), θ_{min}, r_{min}

5) Evaluate.

$$\text{Inner} \int = \int_0^{2\cos\theta} (2r\cos\theta - r^2)\, r\,dr$$

$$= \int_0^{2\cos\theta} (2r^2\cos\theta - r^3)\,dr = \frac{2}{3}r^3 \Big|_0^{2\cos\theta} \cdot \cos\theta - \frac{r^4}{4}\Big|_0^{2\cos\theta}$$

$$= \frac{2\cdot 8}{3}\cos^3\theta \cdot \cos\theta - \frac{1}{4}\cdot 16\cos^4\theta = \frac{4}{3}\cos^4\theta$$

$$\text{Outer} \int = \frac{4}{3}\cdot \int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta \underset{\uparrow}{=} \frac{4}{3}\int_{-\pi/2}^{\pi/2} \Big(\underbrace{\frac{1+\cos 2\theta}{2}}_{\cos^2\theta}\Big)^2 d\theta$$

Sec. 7.2, Ex. 4 (half-angle formula)

$$= \frac{4}{3}\cdot\frac{1}{4}\int_{-\pi/2}^{\pi/2} (1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\text{half-angle f-la again}})\,d\theta$$

$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta \equiv \frac{1}{3}(I_1 + I_2 + I_3).$$

$$I_1 = \int_{-\pi/2}^{\pi/2} \frac{3}{2}\,d\theta = \frac{3}{2}\,\theta\Big|_{-\pi/2}^{\pi/2} = \frac{3}{2}\cdot\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}.$$

For I_2 and I_3:

$$\int \cos(a\theta)\,d\theta = \int \cos u \cdot \frac{du}{a} = \frac{\sin u}{a} + \cancel{C}$$

u-sub:
$u = a\cdot\theta$
$du = a\cdot d\theta$
$d\theta = \frac{du}{a}$

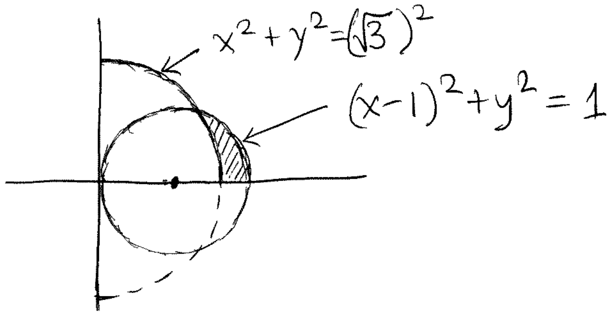
$$I_2 = 2\cdot\frac{\sin 2\theta}{2}\Big|_{-\pi/2}^{\pi/2} = \underbrace{\sin\pi}_{0} - \underbrace{\sin(-\pi)}_{0} = 0$$

$$I_3 = \frac{1}{2}\,\frac{\sin 4\theta}{4}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{8}\left(\underbrace{\sin 2\pi}_{0} - \underbrace{\sin(-2\pi)}_{0}\right) = 0.$$

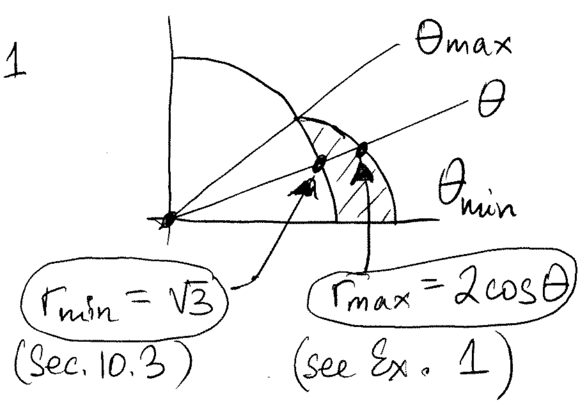
Thus, $V = \frac{1}{3}\left(\frac{3\pi}{2} + 0 + 0\right) = \pi/2.$

Ex. 2 Use polar coord's to set up the area bounded by circles: $x^2+y^2=3$ and $(x-1)^2+y^2=1$ and located in the 1st quadrant.

Sol'n: 0) Sketch



Magnified



1) General formula

Recall from topic 5 of Sec. 15.2 that

$$\text{Area} = \iint_R 1 \cdot dA \longrightarrow r\,dr\,d\theta.$$

2) So we only need to set up limits for R. Use the Magnified sketch.

Inner limits are for r: $\underbrace{\sqrt{3}}_{r_{min}} \le r \le \underbrace{2\cos\theta}_{r_{max}}$

$0 \le \theta \le \theta_{max}$.

@ θ_{max}: $r_{min} = r_{max} \Rightarrow \sqrt{3} = 2\cos\theta \Rightarrow$

$\cos\theta = \sqrt{3}/2 \Rightarrow$ (since need only 1st quadrant) $\theta = \frac{\pi}{6}$

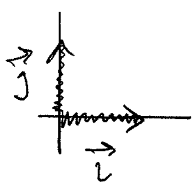
Thus:

$$\text{Area} = \int_0^{\pi/6} \int_{\sqrt{3}}^{2\cos\theta} r\,dr\,d\theta.$$

Note (illustrated by both Ex. 1 & 2)

If a problem gives you limits in Cartesian coord's, then: 0. Sketch (you think in Cartesian, not polar, so that's why you sketch from Cartesian);
1. set up the polar limits from your sketch.

② Unit vectors in polar coordinates



Cartesian:

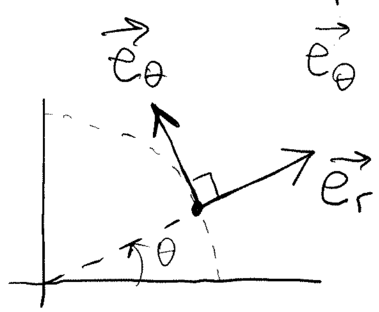
$\vec{i}$: only x changes along $\vec{i}$

$\vec{j}$: — y —— $\vec{j}$

Polar:

$\vec{e_r}$: only r changes along $\vec{e_r}$

$\vec{e_\theta}$: only θ —— $\vec{e_\theta}$



The unit coord. vectors depend on the point where they are drawn!

Where have we seen a similar situation earlier?