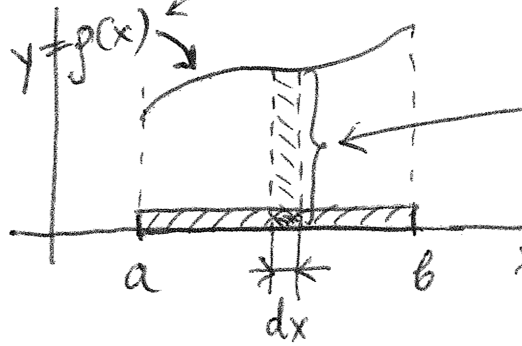
Sec. 15.4 (8th edition).

Some applications of double integrals.

① Mass

Calc. I



Consider a non-uniform beam with mass density $\rho(x)$.

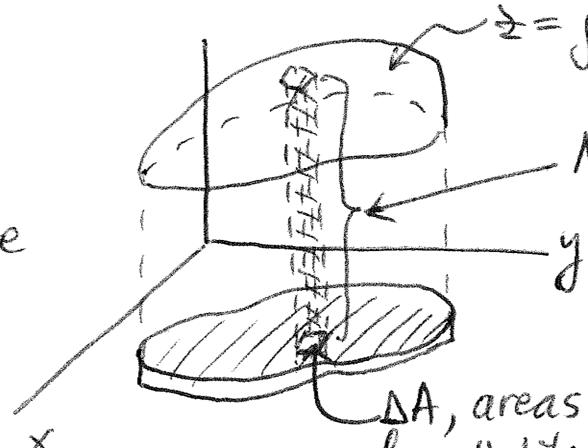
$$\underset{\text{total mass}}{M} = \sum_i M_i \approx \sum_i \overbrace{\rho(x_i)\,\Delta x}^{M_i}$$

As $\Delta x \to 0$,

$$\boxed{M = \int_a^b \rho(x)\,dx}$$

Calc. III

Non-uniform plate (lamina) with mass density $\rho(x,y)$.



$$M = \sum_i M_i = \sum_i \rho(x_i, y_i)\,\Delta A \longrightarrow \iint_R \rho(x,y)\,dA.$$

Thus:

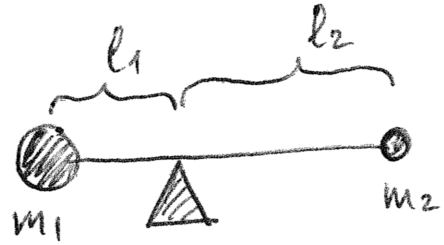
$$\boxed{M = \iint_R \rho(x,y)\,dA}$$

<u>Note</u>: Recall the special case $\rho(x,y)=1$: then

$$\iint_R 1 \cdot dA = \begin{pmatrix}\text{Area} \\ \text{of } R\end{pmatrix}.$$

(2) Center of mass

[a] Motivation.



Consider a lever made of two point-like masses m_1, m_2 connected by a massless rod (as shown).

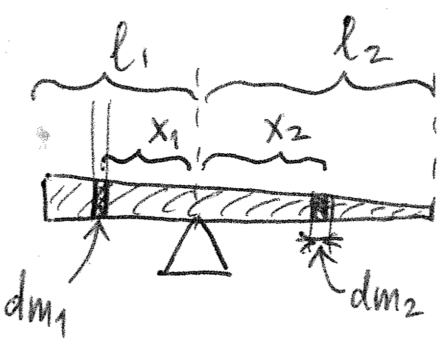
The lever is in balance when:

$$\boxed{m_1 l_1 = m_2 l_2} \qquad (*)$$

("law of the lever", corollary of equation for mechanical torque).

[b] Calc. I

Now consider the problem of balancing a non-uniform beam:



This extended lever will be balanced if every dm_1 on the left of the fulcrum (= the support) is balanced by *some* dm_2 on the right:

$$dm_1 \cdot x_1 = dm_2 \cdot x_2 .$$

Integrating each side, we get:

$$\int_0^{l_1} x_1\, dm_1 = \int_0^{l_2} x_2\, dm_2 , \quad \text{or}$$

$$\boxed{\int_0^{l_1} x_1\, \rho(x_1)\, dx_1 = \int_0^{l_2} x_2\, \rho_2(x_2)\, dx_2 .} \quad (**)$$

Balance condition of an extended lever.

We will <u>not</u> pursue this specific problem, but it *motivates* the concept of *center of mass*:

$$\int_0^{l} x\, \rho(x)\, dx \equiv M \cdot \bar{x}$$

"can be viewed as"

total mass

some "effective" length where all M is lumped.

Then the condition (**) takes on the form:

$$M_1 \cdot \bar{x}_1 = M_2 \cdot \bar{x}_2 ,$$

which is the same form as (*) in $\boxed{a}$.

Returning to

$$\int_0^{\ell} x\,\rho(x)\,dx = M \cdot \bar{x}$$

M: total mass

$\bar{x}$: "effective" location of lumped mass

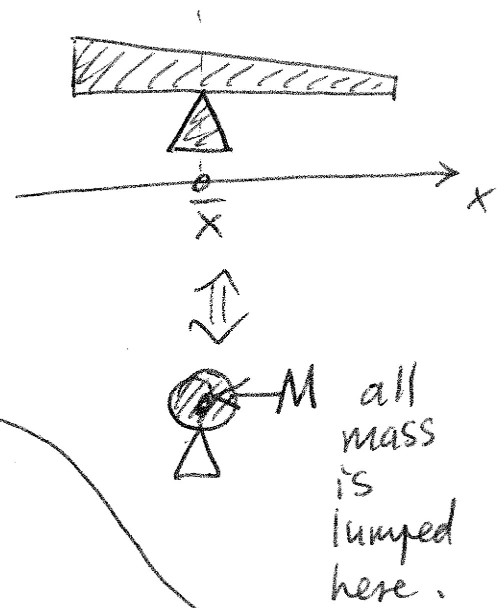
$$\Rightarrow \quad \boxed{\bar{x} = \frac{1}{M}\int_0^{\ell} x\,\rho(x)\,dx}$$

name

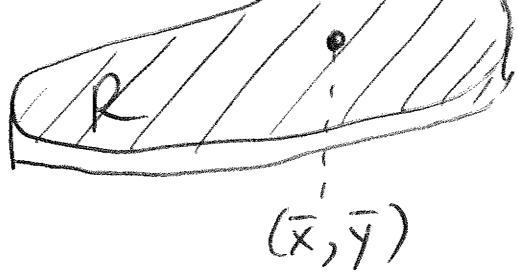
$\bar{x}$: center of mass

(Recall that $M = \int_0^{\ell} \rho(x)\,dx$.)

One can show that if one places the support precisely under $\bar{x}$, then the beam will be in an equilibrium.



c Calc. III



Non-uniform plate R.

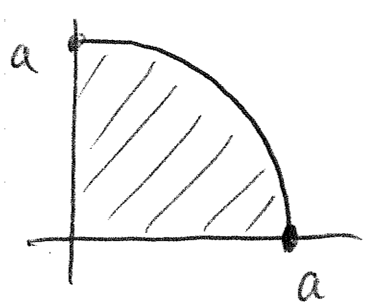
Its $\overbrace{\text{center of mass}}^{\text{CoM}}$ is at $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{1}{M} \iint_R x\, \rho(x,y)\, dA$$

$$\bar{y} = \frac{1}{M} \iint_R y\, \rho(x,y)\, dA.$$

If the support is placed precisely under the CoM, the lamina will be in an equilibrium.

Ex. 1



Find the center of mass of the quarter of a circular plate shown in the figure, whose density is proportional to the distance from the origin.

Sol'n: $\bar{x} = \frac{1}{M} \iint_R x\, \rho(x,y)\, dA$,

$\Rightarrow$ need M first.

1) $M = \iint_R \rho(x,y)\, dA \underset{\substack{\uparrow \\ \text{use polar}}}{=} \int_0^{\pi/2} \left(\int_0^a \underbrace{\overset{\text{const}}{k}\cdot r}_{\rho} \cdot r\,dr \right) d\theta$

$= \int_0^{\pi/2} k \frac{a^3}{3}\, d\theta = \frac{k}{3} a^3 \cdot \frac{\pi}{2}.$

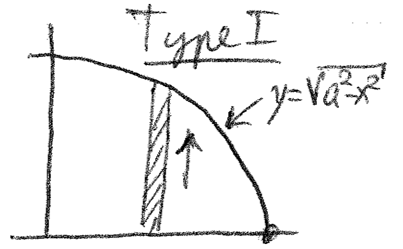
2) We'll do $\bar{x}$ in polar coord's and then $\bar{y}$ in Cartesian, for comparison.

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x,y)\,dA = \frac{1}{M}\int_0^{\pi/2}\left(\int_0^a r\cos\theta\cdot k\cdot r\cdot r\,dr\right)d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2}\cos\theta\cdot\underbrace{\left(\int_0^a kr^3\,dr\right)}_{\frac{ka^4}{4}}d\theta = \frac{1}{M}\cdot\frac{ka^4}{4}\int_0^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{1}{M}\cdot\frac{ka^4}{4}\cdot\underbrace{\sin\theta\Big|_0^{\pi/2}}_{1-0} = \frac{\cancel{k}a^4}{4}\Big/\underbrace{\frac{\cancel{k}a^3\cdot\pi}{6}}_{M}$$

$$= a\cdot\frac{3}{2\pi}\approx\frac{a}{2}$$



3) $$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x,y)\,dx\,dy$$

$$= \frac{1}{M}\int_0^a\left(\int_0^{\sqrt{a^2-x^2}} y\cdot k\cdot\sqrt{x^2+y^2}\,dy\right)dx$$

$u = x^2+y^2$
$du = 2y\,dy$
$u(0) = x^2+0 = x^2$
$u(\sqrt{a^2-x^2}) = x^2+a^2-x^2 = a^2$.

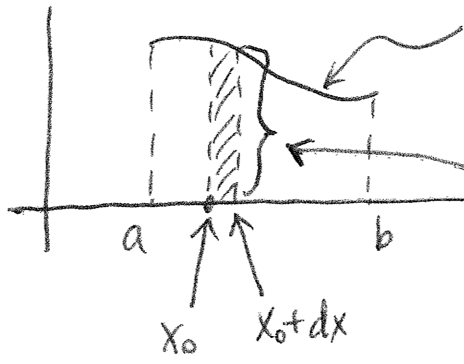
$$= \frac{1}{M}\int_0^a\left(\int_{x^2}^{a^2} k\sqrt{u}\,\frac{du}{2}\right)dx$$

$$= \frac{k}{2M}\int_0^a \frac{u^{3/2}}{3/2}\Big|_{x^2}^{a^2} = \frac{k}{M\cdot 3}\int_0^a (a^3-x^3)\,dx$$

$$= \frac{k}{3M}\left(a^4-\frac{a^4}{4}\right) = \frac{ka^4}{4M} = \bar{y}$$ ← as expected.

3 Probability

Calc. I



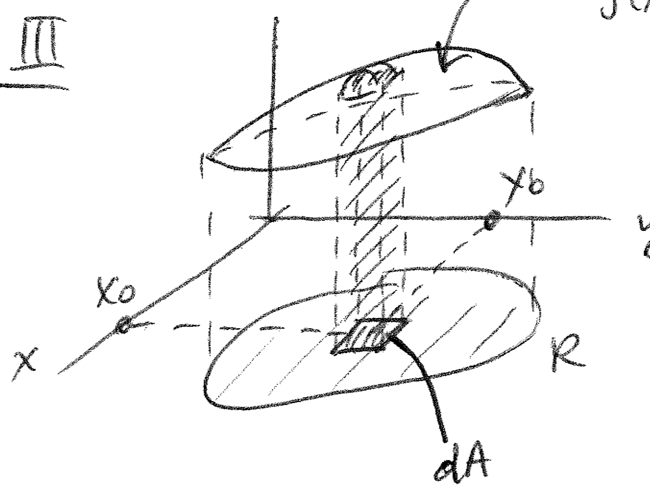
$f(x)$ = probability density function (PDF)

Meaning of PDF: $f(x_0)\,dx$ = probability that X is found inside $[x_0, x_0+dx]$.

Then

$$\boxed{\underbrace{P(a \le x \le b)}_{\text{probability that } x \text{ is in } [a,b]} = \underbrace{\int_a^b f(x_0)\,dx}_{\text{sum of all these probabilities}}}$$

Calc III



$f(x,y)$ = joint PDF of X & Y.

$$\underbrace{f(x_0,y_0)\,dA}_{\text{"volume of the column above } dA\text{"}} = \text{probability that } X \,\&\, Y \text{ are found inside the small tile } dA \text{ near } (x_0,y_0).$$

$$\iint_R p(x,y)\,dA = \begin{pmatrix}\text{probability}\\ \text{that } x \,\&\, y\\ \text{are inside } R\end{pmatrix}.$$

<u>Ex. 2</u> The joint PDF for (X,Y) is:

$$f(x,y) = \begin{cases} cxy, & (x,y) \text{ is in } R=[0,2]\times[0,3] \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the constant C that makes $f(x,y)$ a PDF.

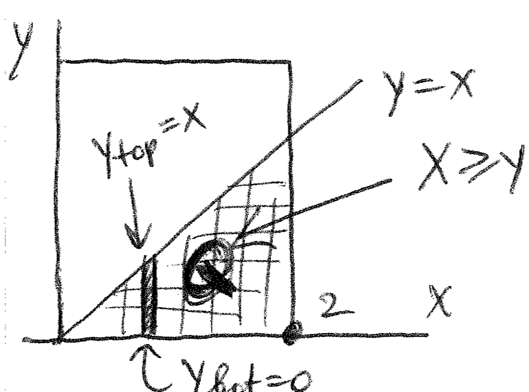
(b) Find $P(X \geq Y)$.

<u>Sol'n:</u> (a) $\begin{pmatrix}\text{Probability that}\\ (x,y) \text{ is } \underline{\text{somewhere}}\\ \text{in } R\end{pmatrix} = 1.$

$$\iint_R Cxy \cdot dA = 1 \quad \leftarrow \text{ Find } C \text{ from this condition.}$$

$$1 = \int_0^3 \left(\int_0^2 Cx\,dx \right) y\,dy = C\int_0^2 x\,dx \int_0^3 y\,dy = C \cdot \frac{2^2}{2} \cdot \frac{3^2}{2}$$

$$\Rightarrow C = 1/9.$$



(b) $P(X \geq Y)$

$$= \iint_Q f(x,y)\,dx\,dy$$

$$= \int_0^2 \left(\int_0^x \tfrac{1}{9} xy\,dy \right) dx =$$

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$$= \int_0^2 \left(\frac{x}{9} \cdot \frac{y^2}{2} \Big|_0^x \right) dx = \int_0^2 \frac{x}{9} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{1}{18} \cdot \frac{x^4}{4} \Big|_0^2 = \frac{2}{9} \, .$$ ← answer for (b).

<u>HW</u>: $\underbrace{3, 9, 16}_{\text{mass, CoM}}; \underbrace{27}_{\text{probability}}$.

non/WA