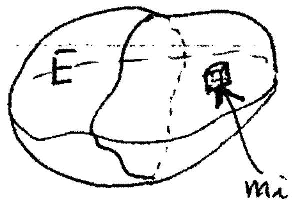
Sec. 15.6. Triple integrals

① Motivation and basic definition



m_i

Suppose we want to find the mass of a nonuniform solid E.

$$M = \sum_i m_i \,, \qquad m_i = \underbrace{\rho}_{\text{density}}(x_i, y_i, z_i)\,\underbrace{dxdydz}_{\Delta V}$$

$dxdydz$ = volume of a small box around (x_i, y_i, z_i)

So

$$M = \lim_{\Delta V \to 0} \sum_i \rho(x_i, y_i, z_i)\,\Delta V$$

$$= \boxed{\iiint_E \rho(x, y, z)\,dxdydz} \leftarrow \text{triple } \textstyle\int \text{ over } E.$$

Other examples:

- Volume of $E = \sum_i \Delta V = \iiint_E 1 \cdot dV \equiv \iiint_E dxdydz$.

- Similar to Sec. 15.4, we can define x, y, z-coord's of the center of mass of E:

  $$\bar{x}_c = \iiint_E x\,\rho(x, y, z)\,dV \,/\, M \qquad \left(M = \iiint_E \rho(x, y, z)\,dV\right).$$

  and similar for $\bar{y}_c$, $\bar{z}_c$.

- If $p(x, y, z)$ is the joint probability density of quantities x, y, z, then

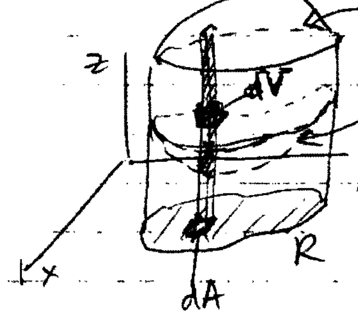
  $\iiint_E p(x, y, z)\,dV$ = Probability that the simultaneous event (x, y, z) is found within bounds of E.

- (lower-case sigma) If $\sigma(x, y, z)$ is the density of electric charge of solid E, then $Q = \iiint_E \sigma(x, y, z)\,dV$ = total electric charge of E.

(2) Computing a $\iiint$ as an iterated integral.

Suppose solid E is bounded by surfaces $z_{bot} = g_1(x,y)$, $z_{top} = g_2(x,y)$ and its projection on xy-plane is a region R. Then:

$$\iiint_E f(x,y,z)\, dV$$



$z_{top} = g_2(x,y)$

$z_{bot} = g_1(x,y)$

$dV = \underbrace{dz} \cdot \underbrace{dA}$

$$\boxed{\iiint_E f\, dV = \iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right) dA}$$

Thus, we reduced the problem of computing a triple int'l to two familiar steps:

1) Compute the single int'l over z;
2) Compute the double int'l over R.

See in book: Ex 1, 2, 4, 5.

Often E can be viewed alternatively, e.g., as bounded by surfaces $y_{bot} = h_1(x,z)$, $y_{top} = h_2(x,z)$ and having a projection S on the xz-plane. (This is analogous to how one can view some regions in the xy-plane as Type I & Type II; Sec. 15.2.)

Then, if $f(x,y,z)$ is continuous over E, one has:

$$\iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right) \underbrace{dA}_{\text{in }(xy)} =$$

in xy-plane $\to R$

$$\iint_S \left(\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\, dy \right) \underbrace{dA}_{\text{in }(xz)} = \iint_T \left(\int_{\cdots}^{\cdots} f(x,y,z)\, dx \right) \underbrace{dA}$$

in xz-plane $\to S$

$T \leftarrow$ in $(yz) \to$

<u>Ex-1</u> Describe the solid E defined by the integration limits. Then rewrite the given $\iiint$ as an iterated $\int$ in all other possible orders.

$$I = \int_{-2}^{2}\int_{-1}^{3-y^2}\int_{0}^{1+z} f(x,y,z)\,dx\,dz\,dy$$

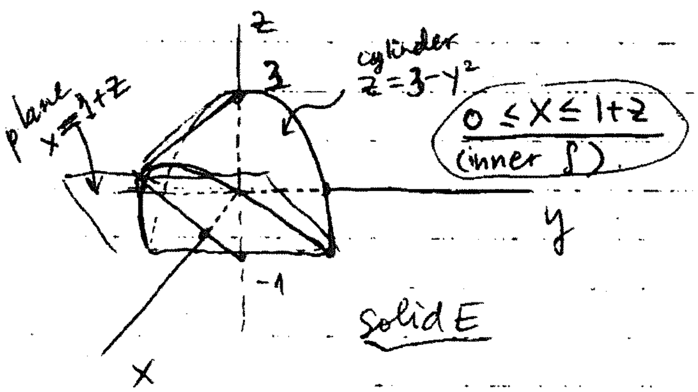
<u>Note 1</u>: Book's notation for solid $E = \{0 \le x \le 1+z,\ -1 \le z \le 3-y^2,\ -2 \le y \le 2\}$.
<u>Note 2</u>: # of permutations of dx, dy, dz is: $3 \cdot 2 \cdot 1 = 6$.

So we need to rewrite I in $5\ (=6-1)$ alternative orders.

<u>Sol'n:</u> 1) Describe E. <u>Sketch it first.</u>

Recall that
$$\iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dx\,dy$$
defines a solid bounded by $z_{bot} = g_1(x,y)$, $z_{top} = g_2(x,y)$ over region R in xy-plane.

By analogy, solid E defined by the limits of I is:
- bounded by $x_{bot} = 0$ & $x_{top} = 1+z$ (inner limits);
- over some region T in yz-plane.



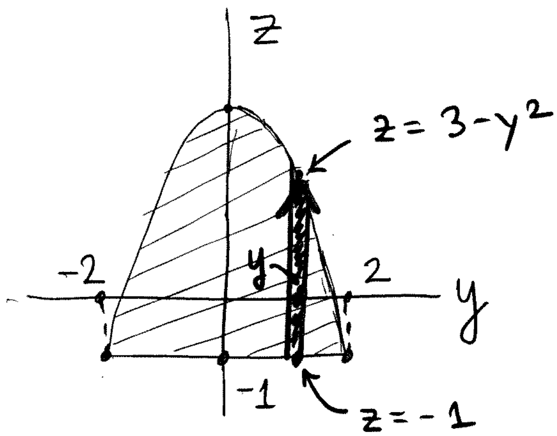
→ Thus: when solving problems about triple integrals, one must SKETCH <u>not only</u> the 3D object, <u>but also</u> its projection on one of the coord. planes.

We will practice doing so in this Example.

! $\boxed{\text{So, remember: You need 2 SKETCHes for } \iiint \text{, not just 1.}}$

We now sketch region T based on the limits of the (z,y)-integration and the 3D sketch.

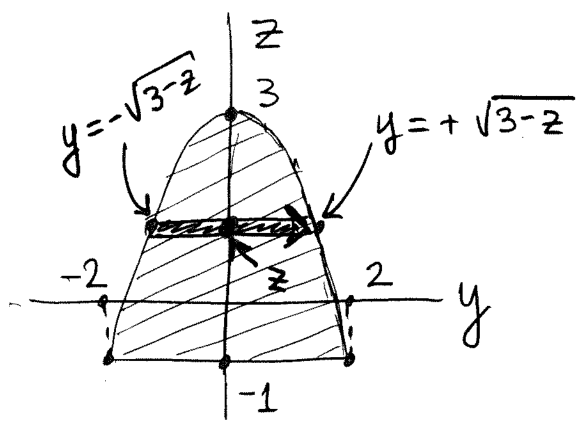
$$I = \int_{-2}^{2} \int_{-1}^{3-y^2} \left(\int_0^{1+z} f \, dx \right) dz \, dy \Rightarrow -1 \le z \le 3-y^2$$
$$-2 \le y \le 2$$



Region T set up as Type I.

We now need to set up the limits in the remaining 5 ways.

2) (a) Region T in the yz-plane as Type II



Region T as Type II.

Inner limits:
$$-\sqrt{3-z} \le y \le \sqrt{3-z}$$

Outer limits:
$$-1 \le z \le 3$$

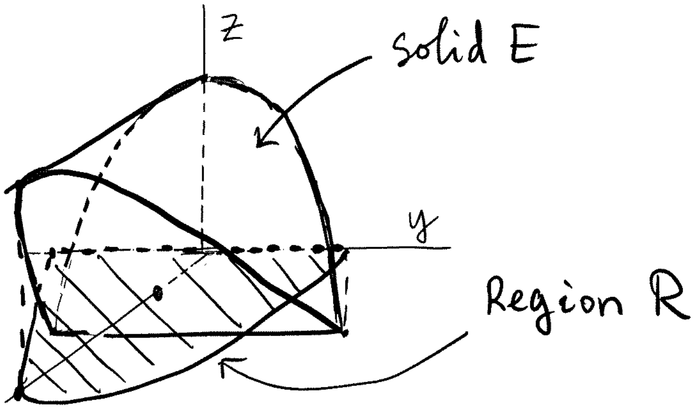
So $I =$

$$\int_{-1}^{3} \int_{-\sqrt{3-z}}^{\sqrt{3-z}} \left(\int_0^{1+z} f \, dx \right) dy \, dz$$

(b) Project solid E on region R in the xy-plane:

We no longer can manipulate the original limits.

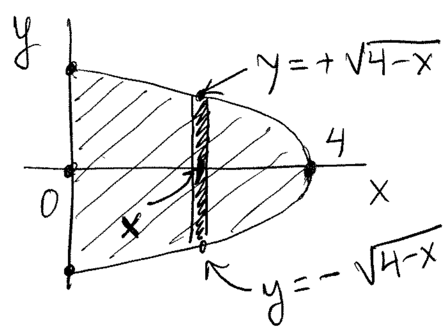
We MUST USE THE 3D SKETCH to find new limits.



To find R, we need to find the projection of the intersection curve of $z = 3 - y^2$ & $x = 1 + z$ on the xy-plane. From the 3D sketch, we see that $z_{top} = 3 - y^2$, $z_{bottom} = x - 1$, so we are setting up $z_{top} = z_{bot}$ (as in Ex. 1 in Sec. 15.3) to find one of the boundaries of R:

$$3 - y^2 = x - 1 \Rightarrow x = 4 - y^2 \text{ (sideways parabola).}$$

The other boundary of R is $x = 0$ (see the 3D sketch).



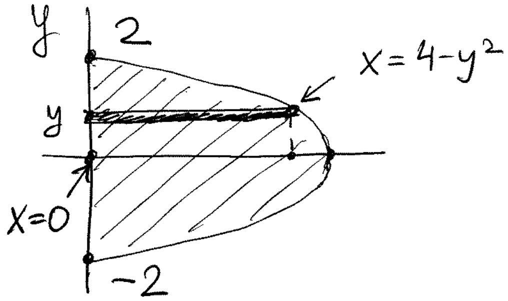
R as Type I

$$-\sqrt{4-x} \le y \le \sqrt{4-x}\ ; \quad 0 \le x \le 4$$

$$\Rightarrow$$

$$I = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} \left(\int_{\underbrace{x-1}_{z_{bot}}}^{\overbrace{3-y^2}^{z_{top}}} f\,dz \right) dy\,dx$$

Now, R as Type II:



$$0 \le x \le 4 - y^2, \quad -2 \le y \le 2$$

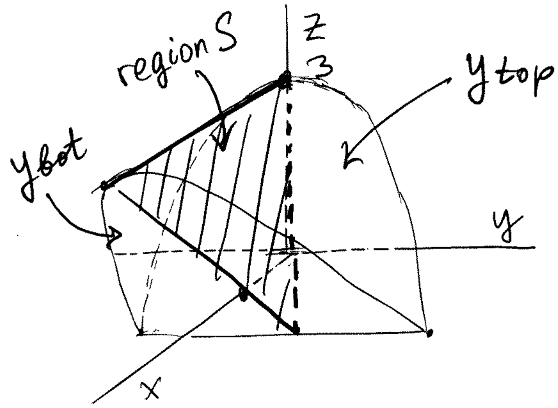
$$\Rightarrow$$

$$I = \int_{-2}^{2} \int_0^{4-y^2} \left(\int_{x-1}^{3-y^2} f\,dz \right) dx\,dy$$

23-6

(c) Project solid E on region S in xz-plane

Again, we look at the 3D sketch to identify y_{top} & y_{bot}; these come from cylinder $z = 3 - y^2$:



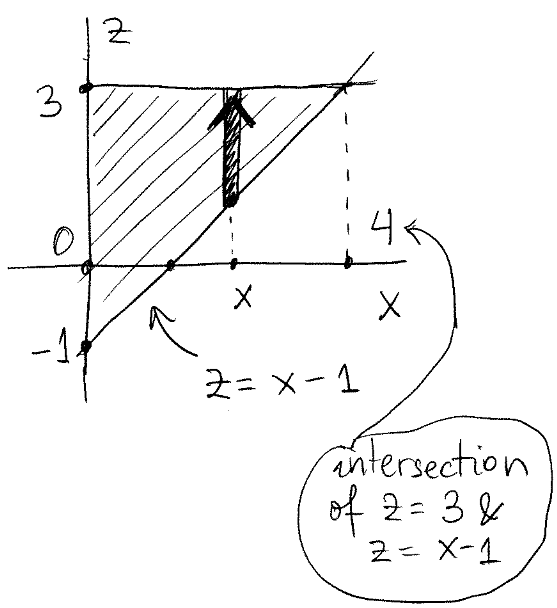
$$y_{top} = +\sqrt{3-z}$$
$$y_{bot} = -\sqrt{3-z}$$

Boundary of S comes from

$$y_{top} = y_{bot}$$
$$+\sqrt{3-z} = -\sqrt{3-z} \Rightarrow$$

$z = 3$ (could have been found by looking at the sketch)

The other two boundaries of S come from the two planes perpendicular to the xz-plane: $x = 0$ & $z = x - 1$.



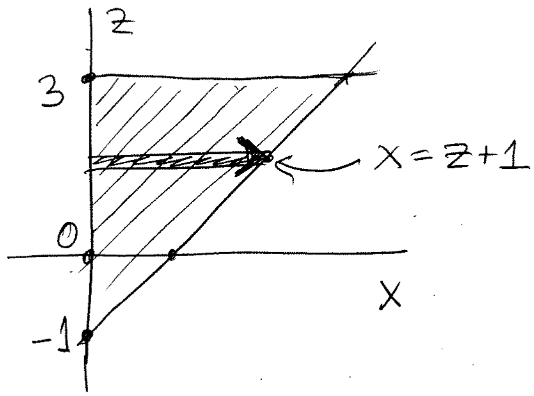
Region S as Type I:

$$x - 1 \le z \le 3, \quad 0 \le x \le 4$$

So,

$$I = \int_0^4 \int_{x-1}^{3} \left(\int_{\underbrace{-\sqrt{3-z}}_{y_{bot}}}^{\overbrace{\sqrt{3-z}}^{y_{top}}} f\, dy \right) dz\, dx$$

Region S as Type II:



$$0 \le x \le z+1, \quad -1 \le z \le 3$$

$$I = \int_{-1}^{3} \int_{0}^{z+1} \left(\int_{-\sqrt{3-z}}^{\sqrt{3-z}} f \, dy \right) dx \, dz$$