(3) And more applications... : Moment of inertia of a solid object.

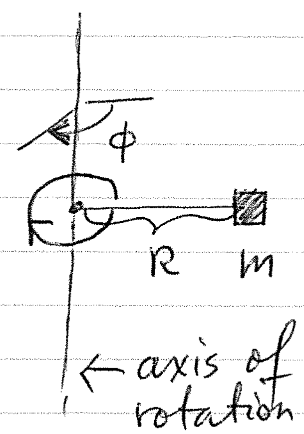
In Sec. 13.4 we showed that the 2nd Law of Newton for the rotational motion can be written as:

$$m R^2 \cdot \frac{d^2\phi}{dt^2} = M \qquad (*)$$

$$\begin{pmatrix}\text{moment of}\\ \text{inertia}\end{pmatrix} \cdot \begin{pmatrix}\text{angular}\\ \text{acceleration}\end{pmatrix} = (\text{torque})$$

$\boxed{\text{(NOT "mass")}}$

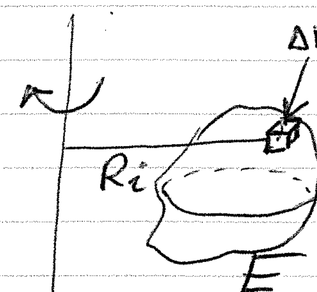
This was for a point-like mass m.



Q: What if we have an extended object?

Follow the usual procedure:

1) Subdivide E into small volumes, with mass ΔM.



2) Apply (*) to each of these small volumes:

$$\sum_i (\Delta M_i R_i^2) \cdot \underbrace{\frac{d^2\phi}{dt^2}} = \sum_i M_i$$

IMPORTANT: this $\frac{d^2\phi}{dt^2}$ is the same for all points of the rotating object. So — factor it.

Replace $\sum_i$ with $\iiint_E$ and get:

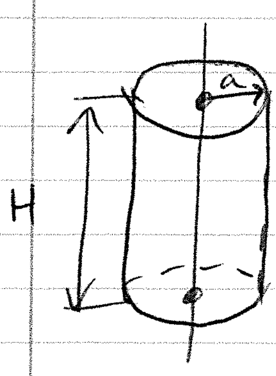
$$\Big(\iiint_E \underbrace{R^2(x,y,z)}_{\substack{\text{distance to}\\ \text{the axis of}\\ \text{rotation}}}\ \underbrace{dm}_{\rho(x,y,z)\,dV}\Big)\cdot\frac{d^2\varphi}{dt^2} = \iiint_E \underbrace{dM(x,y,z)}_{\vec{r}(x,y,z)\times d\vec{F}}$$

__Thus:__

$$\underbrace{\boxed{\Big(\iiint_E \rho(x,y,z)\cdot R^2(x,y,z)\,dV\Big)}}_{\text{"}I\text{", moment of inertia of an extended object.}}\cdot\frac{d^2\varphi}{dt^2} = \iiint_E dM$$

notation

E.g., for a uniform cylinder ($\rho \equiv \overset{\text{const}}{\rho_0}$):



$$I = \iiint_E \rho_0\cdot r^2\,dV$$

$$= \rho_0\cdot\int_0^H\Big(\int_0^{2\pi}\Big(\int_0^a r^2\cdot r\,dr\Big)d\theta\Big)\,dz$$

$$= \rho_0\cdot\int_0^H\int_0^{2\pi}\frac{a^4}{4}\,d\theta\,dz$$

$$= \frac{1}{2}\underbrace{\rho_0 H\pi \underbrace{a^4}_{a^2\cdot a^2}}_{m_{cylinder}} = \frac{1}{2}\,m\cdot a^2.$$

Moments of inertia of other objects will be considered in your engineering courses.