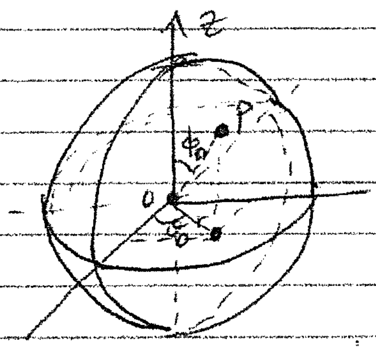
Sec. 15.8. Triple integrals in spherical coordinates

① Spherical coordinates

Pick a point P on a sphere $x^2+y^2+z^2=a^2$ and specify its location using cartesian and cyl. coord's:



Cartesian: $(x_0, y_0, \sqrt{a^2-x_0^2-y_0^2})$

Cylindrical: $(r_0, \theta_0, \sqrt{a^2-r_0^2})$.

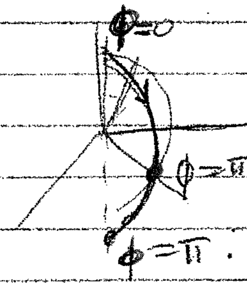
The presence of a $\sqrt{\ }$ indicates that neither coord's are "natural" for a sphere.

Introduce:

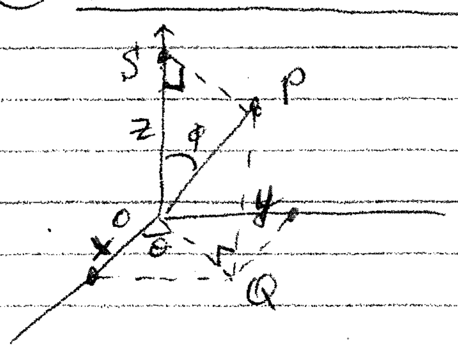
spherical coord's of P $\begin{cases} \rho = a & \leftarrow \text{distance } |OP| \\ \theta = \theta_0 & \leftarrow \text{azimuth} \\ \phi = \phi_0 & \leftarrow \text{polar angle (between } z\text{-axis } \& \ \overrightarrow{OP}\text{)}. \end{cases}$

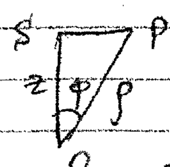
Limits of (ρ, θ, ϕ):

$$0 \le \rho \le \infty$$
$$0 \le \theta \le 2\pi$$
$$\boxed{0 \le \phi \le \pi}$$

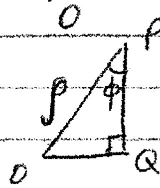


② Relation between spherical and cartesian coord's



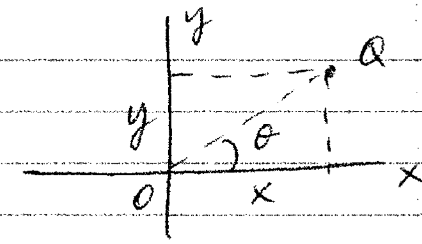


$z = \rho\cos\phi$



$|OQ| = \rho\sin\phi$

View in xy-plane:



$x = |OQ|\cos\theta = \rho\sin\phi\cos\theta$

$y = |OQ|\sin\theta = \rho\sin\phi\sin\theta.$

Thus:

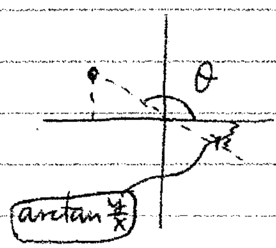
$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}}$$

<u>Conversely:</u>

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\tan\theta = \frac{y}{x} \quad \text{(same as in polar coord's)}$$

$$\phi = \arccos\left(\frac{z}{\rho}\right).$$



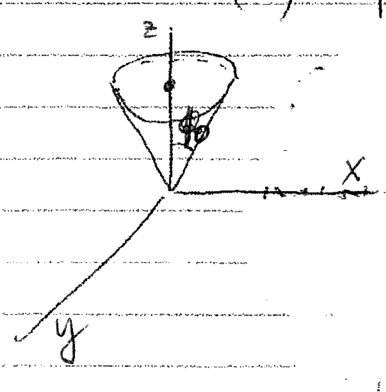
$$\boxed{\begin{aligned} \theta &= \arctan(y/x), \quad x>0 \\ \theta &= \arctan(y/x)+\pi, \quad x<0 \end{aligned}}$$

See Ex. 1, 2 in book.

3 <u>Surfaces defined in spherical coordinates.</u>

<u>Ex. 1</u> (a) $\rho = \rho_0$ (sphere $x^2+y^2+z^2=\rho_0^2$)

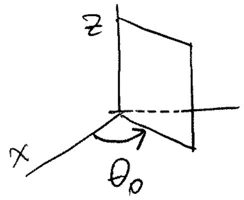
(b) $\phi = \phi_0$ looks like a cone....



<u>In Ex. 2 below,</u> we will:

1) Confirm this, and
2) Show how to find ϕ_0 from the Cartesian equation of the cone.

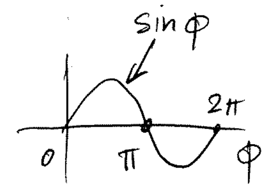
(c) $\theta = \theta_0$ Vertical half-plane making angle θ_0 with the xz-plane (same as in cylindrical coord's).



(d) Cylinder $x^2 + y^2 = b^2$

$$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = b^2$$

$$(\rho\sin\phi)^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = b^2$$



$(\rho\sin\phi)^2 = b^2 \Rightarrow (\rho\sin\phi) = \pm b$.

But $0 \le \phi \le \pi \Rightarrow \sin\phi \ge 0 \Rightarrow$ "$-$" is not possible.

So: spherical equation $\boxed{\rho\sin\phi = b}$ describes the cylinder $x^2 + y^2 = b^2$.

4 Elementary volume dV in spherical coord's

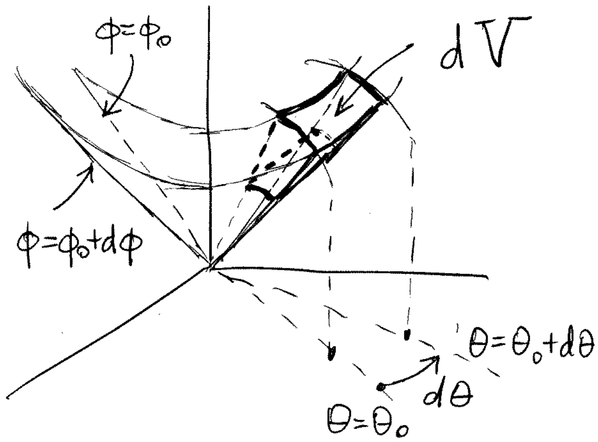
- Goal/motivation: When in $\iiint_E f(x,y,z)\,dxdydz$, the solid E has elements of spherical symmetry, we want to evaluate the integral in spherical coord's.

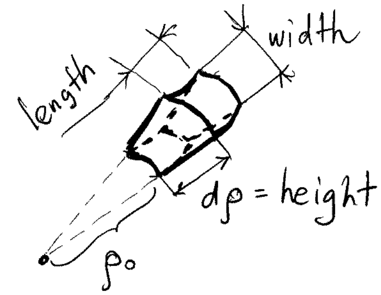
I.e., we seek:

$$\iiint_E f(x,y,z)\underbrace{(dxdydz)}_{dV \text{ in Cartesian}} = \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underbrace{dV}_{? \text{ in spherical}}$$

In full analogy with Cartesian and polar coordinates (see the beginning of Sec. 15.3), the dV in spherical is the volume of a small region bounded by surfaces:

$$\rho_0 \le \rho \le \rho_0 + d\rho, \quad \phi_0 \le \phi \le \phi_0 + d\phi, \quad \theta_0 \le \theta \le \theta_0 + d\theta$$



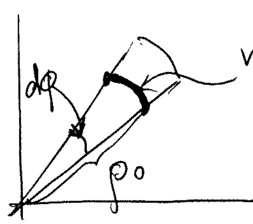


magnified dV
(see also Figs. 7, 8 in textbook)

The elementary volume shown above looks approximately as a rectangular box; the similarity gets stronger as the box gets smaller. So,

$$dV \approx \underbrace{\text{height}}_{d\rho} \cdot \underbrace{\text{length}}_{?} \cdot \underbrace{\text{width}}_{?}$$

Side view to find width:

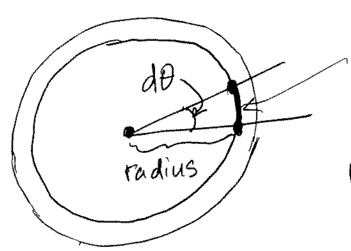


width: arclength = radius $\cdot$ (subtended angle)

$\boxed{\text{width} = \rho_0 \cdot d\phi}$

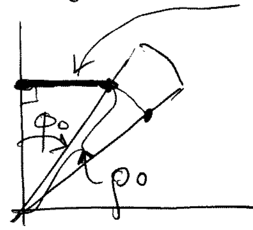
(same as for polar; see p. 21-2)

Top view to find length:



length = radius $\cdot\, d\theta$

To find the "radius", we need the side view again:



radius = $\rho_0 \cdot \sin\phi_0$

thus, $\boxed{\text{length} = \rho_0 \cdot \sin\phi_0 \cdot d\theta}$

Putting all together:

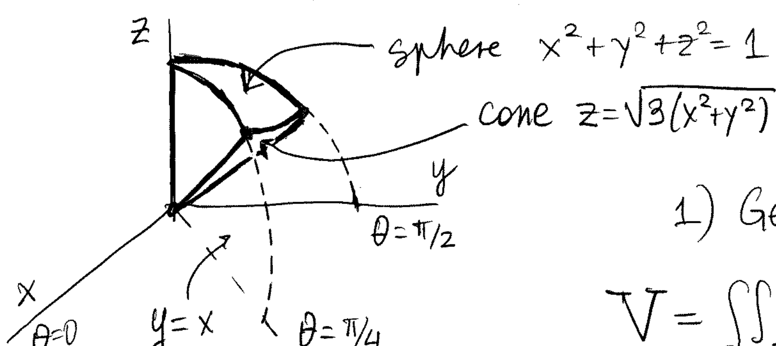
$dV = \text{height} \cdot \text{width} \cdot \text{length} = d\rho \cdot \rho_0 d\phi \cdot \rho_0 \sin\phi_0 \, d\theta \Rightarrow$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta} \leftarrow \text{MUST MEMORIZE}$$

<u>Ex. 2</u> Find the <u>volume</u> of a spherical sector (a "watermelon wedge") in spherical coords. The wedge is bounded above by sphere $x^2+y^2+z^2=1$, below by the top-half of the cone, $z=\sqrt{3(x^2+y^2)}$, and by planes $y=x$ and $x=0$ on the sides.

<u>Note</u>: In this Example we will also learn how to find "ϕ_0" in the equation of a cone (see end of p. 25-2).

<u>Sol'n</u>: 0) <u>SKETCH</u>



1) General formula:

$$V = \iiint_E \underbrace{dV}_{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta} \quad \left(\text{see p. 25-1}\right)$$

2) Need limits for ρ, ϕ, θ.

(a) ρ; distance from origin: $0 \le \rho \le 1$

(from p. 25-2, the eq. of the given sphere is $\rho = 1$).

(b) To find limits for ϕ, we need to transform the cartesian eq. of the half-cone into spherical.

$$z = \sqrt{3} \cdot \sqrt{x^2+y^2} \Rightarrow$$

$$\cancel{\rho}\cos\phi = \sqrt{3} \cdot \cancel{\rho}\sin\phi$$ ← See the work on p. 25-3 for cylinder $x^2+y^2=b^2$.

$$\frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{3}} \Rightarrow \tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

Thus, $0 \le \phi \le \pi/6$ ← on the cone

↑ on the z-axis

(c) The plane $y=x$ goes at $\theta=\pi/4$ to the xz-plane, and plane $x=0$ goes at $\theta=\pi/2$ to the xz-plane.

So: $\pi/4 \le \theta \le \pi/2$.

Combine:

$$V = \int_{\pi/4}^{\pi/2}\int_{0}^{\pi/6}\int_{0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

3) Evaluate:

$$\text{Inner-most} = \int_0^1 \rho^2 d\rho = \frac{\rho^3}{3}\Big|_0^1 = 1/3 \leftarrow \text{const}$$

$$\text{Middle} = \int_0^{\pi/6} \sin\phi \, d\phi = -\cos\phi\Big|_0^{\pi/6} = \left(1-\frac{\sqrt{3}}{2}\right) \leftarrow \text{const.}$$

$$\text{Outer} = \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4.$$

$$V = \frac{\pi}{12}\left(1-\frac{\sqrt{3}}{2}\right)$$

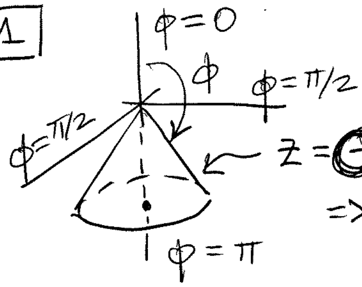
See also Ex. 3 in book.

Now imagine how much more complex this would have been had you used Cartesian coordinates: you would have $\sqrt{\ }$'s in $(z_{top} - z_{bot})$ and also a $\sqrt{\ }$ in the integration limits.

So, use spherical coord's when E has spherical symmetry!

5 Generalizations

1

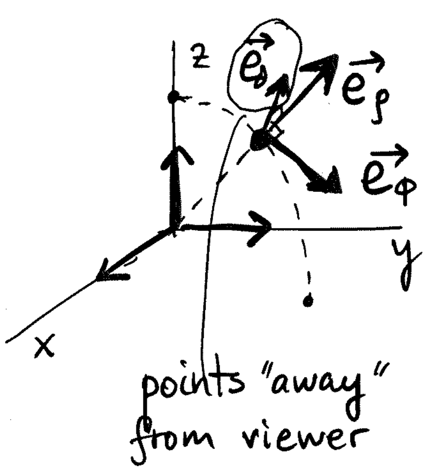


$z = -\text{const}\cdot\sqrt{x^2+y^2}$ $\Rightarrow$ $\boxed{\frac{\pi}{2} \le \phi \le \pi}$

The lower-half of a cone corresponds to

2 **MUST SEE Ex. 4 in book** about a sphere centered on the z-axis & passing through the origin. (This is similar to the circle in Ex. 1 in the Notes for Sec. 15.3; so I recommend that you review that Example, as well as the required Examples from Sec. 10.3, before you study Ex. 4 in the book in Sec. 15.8.)

6 Unit vectors in spherical coordinates



By analogy with topic 2 of Sec. 15.3 (Notes):

$\vec{i}$ = unit vector along the direction where only x changes

$\vec{j}$ = ~~~~~~~~ y ~~~~~~~~

$\vec{k}$ = ~~~~~~~~ z ~~~~~~~~

$\vec{e}_\rho$ = unit vector along direction where only ρ changes

$\vec{e}_\phi$ = ~~~~~~~~~~~~~~~~ ϕ ~~~~~~~~

$\vec{e}_\theta$ = ~~~~~~~~~~~~~~~~ θ ~~~~~~~~

You have seen smth similar in Sec. 13.3...