Sec. 15.9 Change of variables in multiple integrals

1 a Motivation

Consider $\int \cos(x^2)\, 2x\, dx \equiv I$.
Usual way: $u = x^2$, $du = 2x\,dx$, etc.

Equivalent way: $x = \sqrt{u}$, $dx = \frac{dx}{du}\cdot du = \frac{1}{2\sqrt{u}}\, du$, $\Rightarrow$

$$I = \int \cos(u) \cdot \cancel{2\sqrt{u}} \cdot \frac{du}{\cancel{2\sqrt{u}}} = \int \cos u \, du.$$

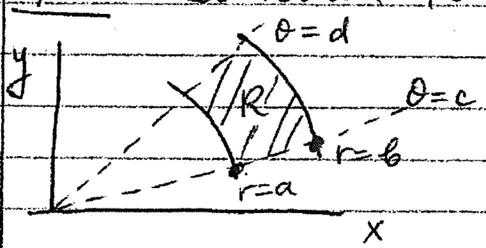
In general: We seek a $x(u)$ such that this integral over u is simpler than the original integral over x.

$$\int f(x)\,dx = \underbrace{\int f(x(u))\, \frac{dx}{du}\cdot du}_{\text{integral over } u \text{ is simpler than the original integral over } x} \quad (*)$$

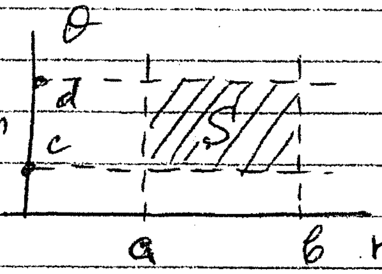
Q: How does this generalize to double integrals?

1 b Example of variable transformation

Ex. 1 Cartesian to polar



$$\underset{\text{Transformation}}{\xleftrightarrow[\ y = r\sin\theta\ ]{x = r\cos\theta}} \qquad \begin{pmatrix} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{pmatrix}$$



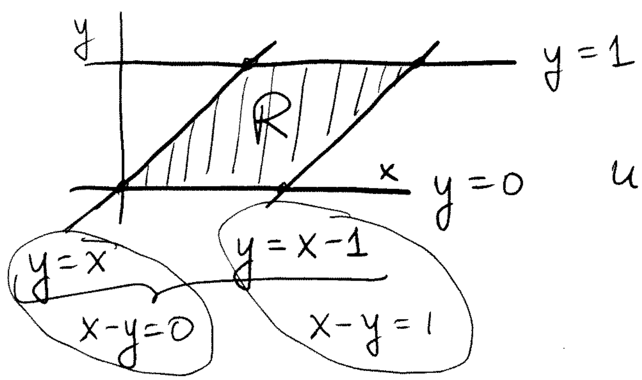
So this transformation takes a non-rectangular region R in xy-plane into a rectangle S in $r\theta$-plane.

$$\iint_R f(x,y)\,(dx\,dy) = \iint_S f(x(r,\theta), y(r,\theta))\; r \; dr\, d\theta \quad (**)$$

In this Section we pursue **2 Goals**:

① Learn how to find a transformation $(x,y) \longleftrightarrow (u,v)$ that transforms **some region R** in the (x,y) plane into a **rectangle in the (u,v)-plane**.

Example 1(a):



Bounds in (x,y):

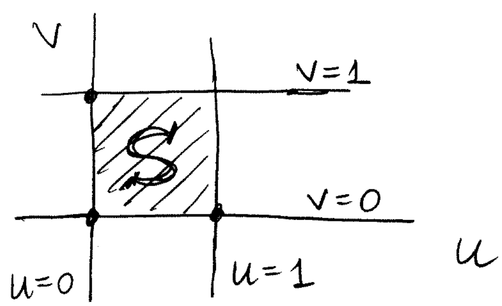
$x - y = 0$, $x - y = 1$
u

$y = 0$, $y = 1$
v

Bounds in (u,v):

$u = 0$, $u = 1$

$v = 0$, $v = 1$



What we will do later is generalize Ex. 1(a) to the case where **R is an arbitrary parallelogram.**

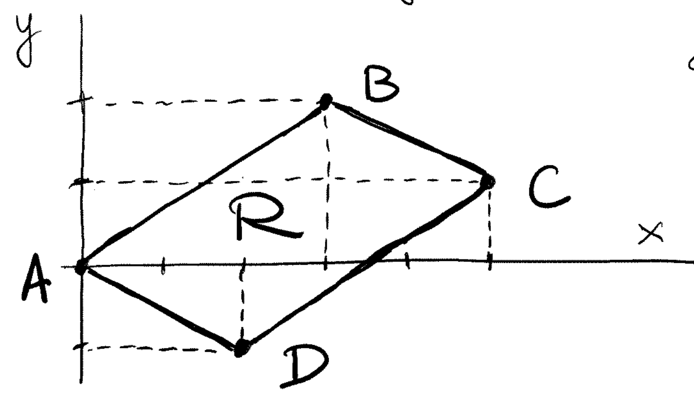
② Once we have found the desired transformation $(x,y) \longleftrightarrow (u,v)$, we will learn how to "transform" $dxdy$, i.e.:

$$\iint_R f(x,y)\,(dxdy) = \iint_S \underbrace{f(x(u,v),\, y(u,v))}_{\text{Some } F(u,v)} \cdot ??? \cdot du\,dv$$

This "???" $= r$ for $(x,y) \leftrightarrow (r,\theta)$, but how do we find it in general?

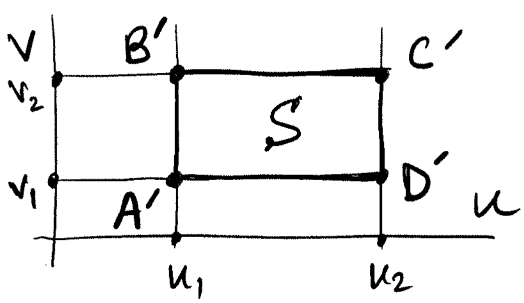
② Achieving Goal 1: Finding a transformation from a parallelogram to a rectangle.

Ex. 2 Find a transformation $(x,y) \leftrightarrow (u,v)$ such that a given parallelogram R in the xy-plane becomes a rectangle in the (u,v)-plane.



Sidenote: Parallelogram R is neither Type I nor Type II region, so integration over it would be complicated.

1) As Ex. 1 & 1a suggest, **the required transformation** $(x,y) \leftrightarrow (u,v)$ **is suggested by the equations of the boundaries of R.**



We seek:

u = const along AB & DC } of R
v = const along AD & BC

(Just for convenience, we will do it in reverse order: first v = const, then u = const.)

v = const along AD: $y = -\frac{1}{2}x$, or $x+2y = 0$

along BC: $y = -\frac{1}{2}x + \text{const}$, or $x+2y = \text{const}$ ← will find later

So $\boxed{v = x + 2y}$

u = const along AB: $y = \frac{2}{3}x$, or $2x - 3y = 0$

along DC: $y = \frac{2}{3}x + \text{const}$, or $2x - 3y = \text{const}$ $\uparrow$ Will find later

$\Downarrow$

$$\boxed{u = 2x - 3y}$$

2) The above work has already given us two boundaries of S': on $A'D'$, $v = 0$ (see the work for "v=const")

on $A'B'$, $u = 0$ (see the work for "u=const")

It remains to find the "const"s for BC & DC.

To do so, pick any point on each of these sides.

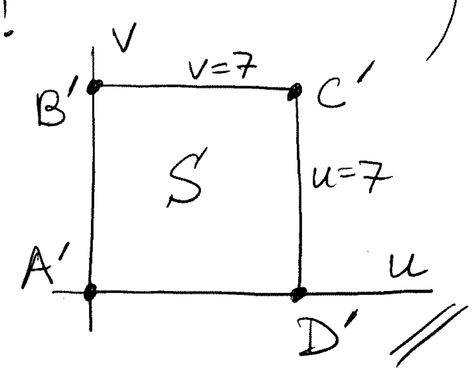
On BC, pick $B = (3,2)$; then $v = (x+2y)\Big|_{@(3,2)} = 3 + 4 = \underline{7}$,

On DC, pick $D = (2,-1)$; then $u = (2x-3y)\Big|_{@(2,-1)} = 4 + 3 = \underline{7}$,

(Note: The fact that these two constants are equal is a mere coincidence!)

Thus, the rectangle S' in (u,v)-plane is:

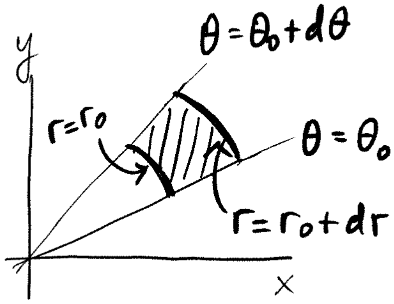
$S = \{0 \le u \le 7,\ 0 \le v \le 7\}$.



Note: We have found $u = x + 2y \equiv u(x,y)$
$v = 2x - 3y \equiv v(x,y)$.

We will see later that we also need the inverse transformation $x = x(u,v)$, $y = y(u,v)$. It is found by solving for x, y: $x = (3u + 2v)/7$, $y = (2u - v)/7$.

③ Achieving Goal 2: Finding dA (= d"area") under the transformation $(x,y) \leftrightarrow (u,v)$.

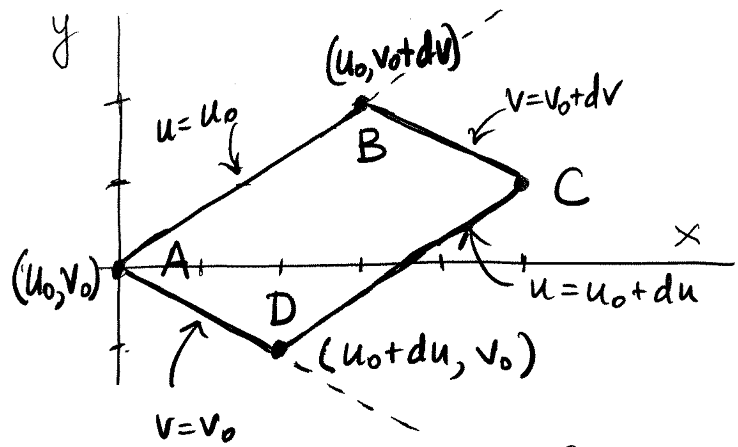


We are again guided by Ex. 1, the change Cartesian → Polar. Namely, there we found dA as the area of a small region bounded by: $r=r_0$, $r=r_0+dr$, $\theta=\theta_0$, $\theta=\theta_0+d\theta$.

In complete analogy, we will find dA as the area of a small parallelogram bounded by:

$u=u_0$, $u=u_0+du$

$v=v_0$, $v=v_0+dv$



1) From Sec. 12.4,

$$S_{ABCD} = |\vec{AD} \times \vec{AB}|$$

So need $\vec{AD}$, $\vec{AB}$ in terms of u,v.

2) $\vec{AD} \equiv \langle x_D, y_D \rangle - \langle x_A, y_A \rangle =$

$$\langle x(u_0+du, v_0), y(u_0+du, v_0) \rangle - \langle x(u_0,v_0), y(u_0,v_0) \rangle$$

Let's focus on the x-coordinate first:

$$x(u_0+du, v_0+0) - x(u_0,v_0) \approx \underline{\frac{\partial x}{\partial u}(u_0,v_0) \cdot du} + \cancel{\frac{\partial x}{\partial v}(u_0,v_0) \cdot 0}^{\,0}$$

And similarly for y. Then:

$$\boxed{\vec{AD} \approx \left\langle \frac{\partial x}{\partial u} \cdot du, \frac{\partial y}{\partial u} \cdot du \right\rangle}$$

Similarly,

$$\vec{AB} = \langle x_B, y_B \rangle - \langle x_A, y_A \rangle$$
$$= \langle x(u_0, v_0 + dv), y(u_0, v_0 + dv) \rangle - \langle x(u_0, v_0), y(u_0, v_0) \rangle$$
$$\approx \langle \frac{\partial x}{\partial v} \cdot dv, \frac{\partial y}{\partial v} \cdot dv \rangle$$

3) Put this into the formula:

$$S_{ABCD} = |\vec{AD} \times \vec{AB}| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u du & y_u du & 0 \\ x_v dv & y_v dv & 0 \end{vmatrix} \right\|$$

length det

$$= \left| \vec{k} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \cdot du\,dv \right| = \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right| \cdot du\,dv$$

det abs. value

<u>Thus:</u>

MUST MEMORIZE !!!

$$\boxed{(dx\,dy) \leftrightarrow \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right| \cdot du\,dv} \quad (J)$$

Change of dA under $(x,y) \leftrightarrow (u,v)$.

<u>Note 1:</u> Need the abs. value, $|\ldots|$, since the area cannot be negative.

<u>Note 2:</u> The quantity J is called **the Jacobian** after the German mathematician Carl Jacobi.

$$J = \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right| \equiv \frac{\partial(x,y)}{\partial(u,v)}$$

Note 3 Rows & columns of J can be switched (this convention is used in the textbook).

Note 4 Recall that in this course, we denote $\vec{r} \equiv \langle x, y \rangle$. Since in this case, $x = x(u,v)$ and $y = y(u,v)$, we have

$$\vec{r} = \langle x(u,v), y(u,v) \rangle \equiv \vec{r}(u,v).$$

Then in our derivation,

$$\vec{AD} = \langle x_u du, y_u du \rangle = \langle x_u, y_u \rangle du = \vec{r}_u \cdot du$$

$$\vec{AB} = \text{(similarly) } \vec{r}_v \cdot dv.$$

I.e., **the side where only u changes** $= \vec{r}_u \cdot du$

the side where only v changes $= \vec{r}_v \cdot dv$

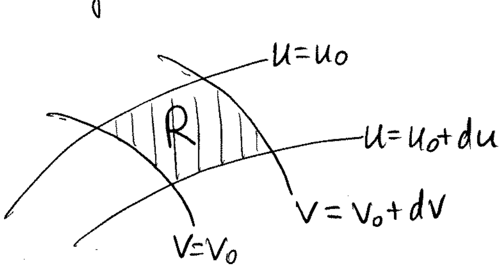
(we will use this later & in Sec. 16.6)

$$\boxed{S_{ABCD} = |\vec{AD} \times \vec{AB}| = |\vec{r}_u \cdot du \times \vec{r}_v dv| = |\vec{r}_u \times \vec{r}_v| \cdot du dv}$$

← Will use this in 16.6 & 16.7.

abs. value

Note 5 If the transformation $(x,y) \leftrightarrow (u,v)$ is such that a region R with curved boundaries (such as the sector in polar coord's; Ex. 1) is transformed into a rectangle S, our derivation does not change, because a very small curved region R is approximately a parallelogram!



Note 6 In the familiar example of changing from Cartesian to polar coordinates,

$$x = \underset{u}{r}\cos\underset{v}{\theta}, \quad y = r\sin\theta, \quad \text{one has:}$$

$$J = \left|\begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix}\right| = \left|\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}\right| =$$

$$\left|\cos\theta \cdot r\cos\theta - \sin\theta \cdot (-r\sin\theta)\right| = \left|r\cos^2\theta + r\sin^2\theta\right| = r,$$

the familiar result.

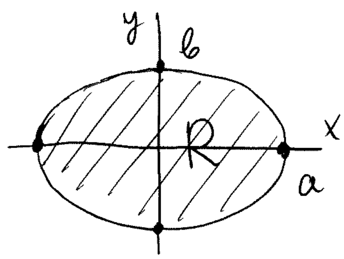
(4) A slight variation: transforming into a circle

We chose to transform a given region R into a rectangle because it is very easy to integrate over a rectangle.

But by now we are also comfortable integrating over a circle (where we can use polar coordinates).

So sometimes it is easier to transform a given region into a circle instead of a rectangle.

Ex. 3(a) Use transformation $x = a \cdot u$, $y = b \cdot v$



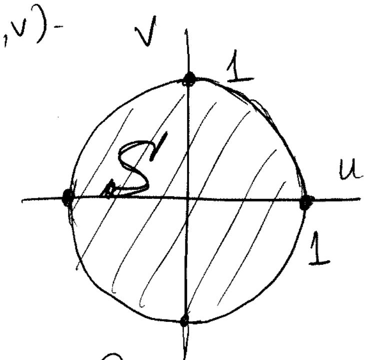
to express $\iint_R f(x,y)\,dx\,dy$ in the (u,v)-coordinates.

<u>Sol'n:</u> 1) Find what R transforms to:

R: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Cartesian eq. of ellipse, Sec. 12.6)

$\Rightarrow S: \frac{(a\cdot u)^2}{a^2} + \frac{(bv)^2}{b^2} = 1 \Rightarrow \frac{\cancel{a^2}u^2}{\cancel{a^2}} + \frac{\cancel{b^2}v^2}{\cancel{b^2}} = 1$

$\Rightarrow \underline{u^2 + v^2 = 1}$ ← circle in (u,v)-plane



2) Change dA:

$dx\,dy = J \cdot du\,dv$ ← Jacobian

$$J = \left|\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}\right| = \left|\begin{vmatrix} \frac{\partial(au)}{\partial u} & \cancel{\frac{\partial(bv)}{\partial u}} \to 0 \\ 0 \leftarrow \cancel{\frac{\partial(au)}{\partial v}} & \frac{\partial(bv)}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}\right|$$

$= a\cdot b, \Rightarrow$

$dx\,dy = ab\cdot du\,dv$

3) $\iint_R f(x,y)\,dx\,dy = \iint_S f(au, bv)\cdot(ab)\cdot du\,dv.$

One can now switch $(u,v) \to (u = r\cos\theta,\ v = r\cdot\sin\theta)$ if need be.

<u>Ex. 3(b)</u> Find the area of the ellipse R from part (a).

<u>Sol'n:</u> $A_R \overset{\text{Sec. 15.2 (end)}}{=} \iint_R 1\cdot dx\,dy \underset{(a)}{=}$

$$= \iint_S 1 \cdot (ab)\, du\, dv = ab \cdot \underbrace{\iint_S 1 \cdot du\, dv}_{\text{area of circle } S}$$

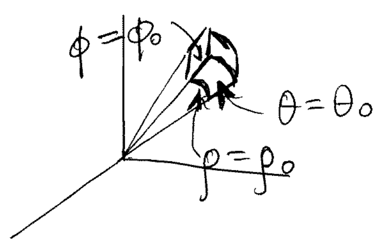
$$= ab \cdot \pi \cdot 1^2 .$$

Thus, $\boxed{A_R = \pi \cdot ab}$.

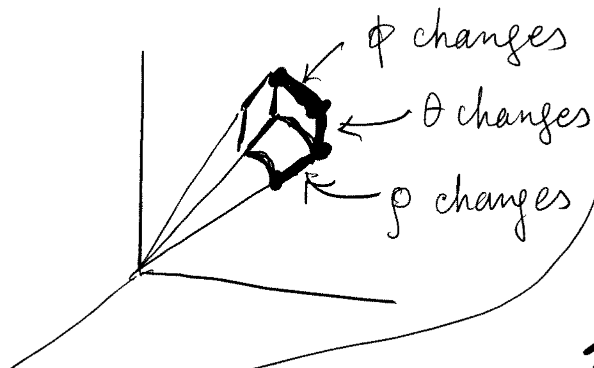
(5) Change of variables in 3D

<u>Question:</u> How does $(dx\,dy\,dz)$ transform under $(x,y,z) \longleftrightarrow (u,v,w)$?

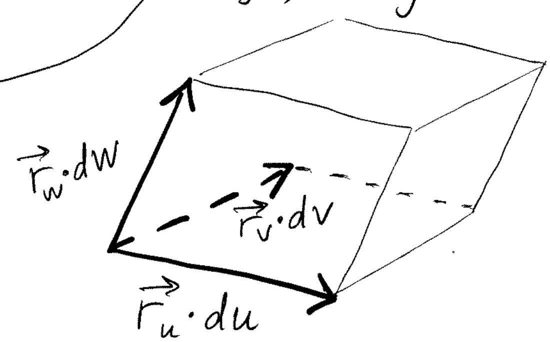
We will proceed by analogy with the change Cartesian $\longrightarrow$ spherical, $(x,y,z) \longleftrightarrow (\rho, \varphi, \theta)$:



We found dV as the volume of a small box, where on <u>edge</u> of the box, only one coordinate would change (ρ, or φ, or θ):



Similarly, in (u,v,w), we consider a box where along each edge, only 1 coordinate changes:



This is similar to Note 4 on p. 26-7:

" **the side where only u changes** $= \vec{r}_u \cdot du$", etc.

Then dV = the volume of this box (Sec. 12.4, triple product)

$$= \left| (\vec{r}_u \cdot du) \bullet (\vec{r}_v\, dv \times \vec{r}_w\, dw) \right| = \left| \underbrace{\vec{r}_u \bullet (\vec{r}_v \times \vec{r}_w)} \right| du\,dv\,dw$$

abs. value

Thus,
$$J = \left| \det \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} \right| \equiv \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|$$

abs. value — det — abs. value

$$dV = dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \cdot du\,dv\,dw$$

For example, when $(u,v,w) \equiv (\rho, \phi, \theta)$, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$

then
$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \equiv \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \rho^2 \sin\phi$$

↑ Ex. 4 in ~~book~~

$$\Rightarrow \quad dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| d\rho\, d\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$