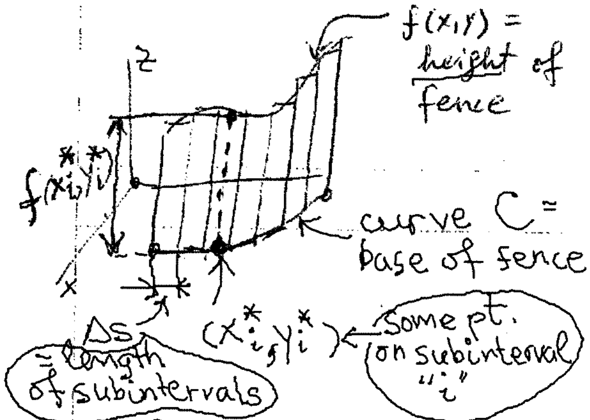
Sec. 16.2. Line integrals

① Line integrals w.r.t. arc length (1st part of Sec. 13.3)

Ex. 1 Find the lateral area of the fence



$$A = \sum_i A_i =$$

$$\approx \sum_i f(x_i^*, y_i^*)\,\Delta s_i \,,$$

$$A = \int_C f(x,y)\,ds.$$

If C: $x = x(t)$, $y = y(t)$, $a \le t \le b$ $\Rightarrow$

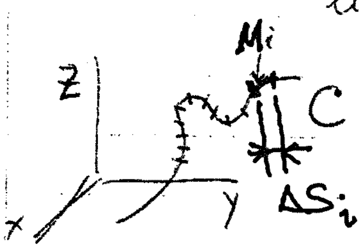
$$A = \int_a^b f(x(t), y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

We derived this in Sec. 13.3 $\rightarrow$ ds (arclength of the "elementary subinterval")

(**MUST REVIEW this formula for ds**)

See Ex. 1, 2 in book for numbers.

Ex. 2 Find the mass of a 3D wire with linear density $\delta(x,y,z)$.



$$M = \sum_i M_i \approx \sum_i \delta(x_i, y_i, z_i)\,\Delta S_i$$

! $$\boxed{M = \int_C \delta(x,y,z)\, ds}$$ We'll interpret the integral w.r.t. arclength as **the mass of a wire.**

If C: $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$ $\Rightarrow$

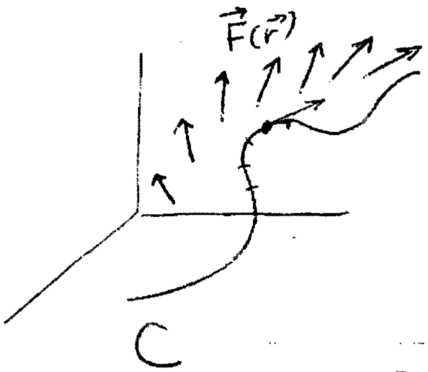
$$M = \int_a^b \delta\left(x(t), y(t), z(t)\right) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}_{ds \text{ in 3D}}.$$

See Ex. 3, 5 in book. (also Sec. 13.3) **MUST REVIEW**

② Line integrals **w.r.t. x, y, z.**

Ex. 3. Find the work done by the force $\vec{F}(\vec{r})$ in moving a particle along the 3D curve C.

Recall the convention: $\vec{F}(\vec{r}) \equiv \vec{F}(x,y,z)$, etc.



$$W = \sum_i W_i$$

$$W_i = \vec{F} \bullet \vec{\Delta r},$$ Formula from physics

where $\vec{\Delta r}$ is the vector along the particle displacement.

But $\vec{\Delta r} = \vec{T} \cdot \Delta s$, where $\vec{T}$ is the unit tangent vector to the curve C. Thus,

$$W_i = (\vec{F} \bullet \vec{T})\, \Delta s, \Rightarrow$$

$$W = \int_C (\vec{F} \bullet \vec{T})\, ds$$

So far, this is the integral of the same form as in Ex. 2.

But we can also look at it differently. Consider again $\vec{F} \bullet \Delta\vec{r}$, but now use the component form of the dot product.

Namely, if $\vec{F}(\vec{r}) = \langle P(\vec{r}), Q(\vec{r}), R(\vec{r}) \rangle$,
and $\Delta\vec{r} = \langle \Delta x, \Delta y, \Delta z \rangle$, $\Rightarrow$

$$\vec{F} \bullet \Delta\vec{r} = P\,\Delta x + Q\cdot\Delta y + R\cdot\Delta z.$$

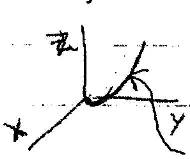
Therefore: We will interprete the integral w.r.t. x, y, z **as work done to move a point on a curve**

$$\boxed{W = \int_C \vec{F} \bullet d\vec{r} = \int_C \left(P(\vec{r})\,dx + Q(\vec{r})\,dy + R(\vec{r})\,dz \right)}$$

$$= I_1 + I_2 + I_3.$$

Thus, to find W, we can compute the three individual integrals I_1, I_2, I_3 and then add them together.

__Ex. 4__ Find the work done by the force $\vec{F} = \langle y, z, -x \rangle$ in moving the particle along the twisted cubic C: $x = 2t$, $y = 3t^2$, $z = t^3$ from $(0,0,0)$ to $(2,3,1)$.



twisted cubic C

__Sol'n:__

$$W = \int_C (P\,dx + Q\,dy + R\,dz) =$$

$$= \int_C (y\,dx + z\,dy - x\,dz)$$

$\boxed{\text{Note: On the curve, } x, y, z \text{ are all dependent, } \Rightarrow \int y\,dx \neq yx \text{ etc.}}$!

$\boxed{\text{Important difference from Chap. 15}}$

$$= \int_{t=0}^{1} 3t^2 \cdot d(2t) + t^3 \cdot d(3t^2) - 2t \cdot d(t^3)$$

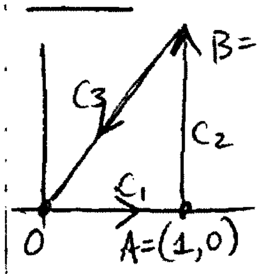
$$= \int_0^1 (3t^2 \cdot 2\,dt + t^3 \cdot 6t\,dt - 2t \cdot 3t^2\,dt)$$

$$= \int_0^1 (6t^2 + 6t^4 - 6t^3)\,dt =$$

$$= \frac{17}{10}.$$

See also Ex. 7, 8 in book.

Ex. 5 Piecewise-smooth path.



If $C = C_1 + C_2 + C_3$, then

$$\int_C \vec{F} d\vec{r} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) \vec{F} d\vec{r}.$$

Find $\displaystyle\int_{C_1+C_2+C_3} \vec{F} d\vec{r}$, $\vec{F} = \langle x^2, x \cdot (y+1) \rangle$.

On C_1:

$0 \le x \le 1$, $y = 0$; $dx = dx$, $dy = 0$.

$$\int_{C_1} x^2 dx + \overbrace{x(0+1) \cdot \underset{dy}{0}}^{0} = \int_0^1 x^2 dx + 0 = \frac{1}{3} + 0.$$

On C_2: $x = 1$, $0 \le y \le 2$; $dx = 0$, $dy = dy$.

$$\int_{C_2} \left(\overset{x^2}{1} \cdot \overset{dx}{0} + \overset{x}{1} \cdot (y+1)\, dy \right) = 0 + \int_0^2 (y+1)\, dy = 0 + 4$$

On C_3: $x: 1 \to 0$ (decreases!) $\Longrightarrow$ $dx = dx$.
$y = 2x$ $\qquad$ $dy = 2dx$.

$$\int_{C_3} x^2 dx + x\,(\underbrace{2x}_{y}+1) \cdot \underbrace{2dx}_{dy}$$

$$= \int_1^0 x^2 dx + (4x^2+2x)\,dx = -\left(\frac{1}{3} + \frac{7}{3}\right).$$

Note this order

because of the order of the integration limits

Thus

$$\int_{C_1+C_2+C_3} \vec{F}\, d\vec{r} = \left(\frac{1}{3}+0\right) + (0+4) + \left(-\frac{1}{3} - \frac{7}{3}\right) = \underline{\underline{\frac{5}{3}}}.$$

See also Ex. 4, 6 in book.

Observation: Look at the first part of this $\int$:

$$\int_{C_1+C_2+C_3} x^2 dx = \frac{1}{3} + 0 - \frac{1}{3} = 0.$$

But this remarkably simple result could have been obtained much quicker! Note that

$$\int_{C_1+C_2+C_3} x^2 dx = \left(\int_{x_{1,start}}^{x_{1,end}} + \int_{x_{2,start}}^{x_{2,end}} + \int_{x_{3,start}}^{x_{3,end}} \right) x^2 dx = \int_{x_{1,start}=0}^{x_{3,end}=0} x^2 dx = 0,$$

because: $x_{1,end} = x_{2,start}\ (=1)$, $x_{2,end} = x_{3,start}\ (=1)$, and we have used the integration properties: $\left(\int_a^b + \int_b^c\right) f(x)dx = \int_a^c f(x)dx$ and $\int_a^a f(x)dx = 0$ for any $f(x)$.

In some HW problems, recognizing that a piece of the integral looks like $\int_C f(x)dx$, or $\int_C g(y)\,dy$, or $\int_C h(z)\,dz$

will reduce the amount of work needed, because

$$\int_C f(u)\,du = \int_{u_{start}}^{u_{end}} f(u)\,du,$$

can be any of x, y, or z

regardless of specific shape of C!

③ Effect of path reversal (will use in Sec. 16.3 & 16.4)

Let "$-C$" denote path C traversed in the opposite direction:

Then:

1) $$\int_{(-C)} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$$

arclength

(the mass of a wire doesn't depend on whether we integrate from A to B or from B to A).

2) $$\boxed{\int_{(-C)} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}}$$!

Will be important in 16.3 & 16.4

or

$$\int_{(-C)} P\,dx + Q\,dy + R\,dz = -\int_C P\,dx + Q\,dy + R\,dz$$

because $d\vec{r}$ on $(-C)$ equals $(-d\vec{r})$ on C (see C_3-part in Ex. 5 above).

Thus, the work we need to do depends on whether we lift an object up or allow it to drop due to gravity.