Sec. 16.3: The Fundamental Theorem for line integrals

① Motivation

The Fundamental Theorem of Calculus (FTC) from Calc. I (Chap. 5):

$$\int_a^b \frac{df(x)}{dx} \cdot dx = f(b) - f(a) \qquad \text{(FTC)}$$

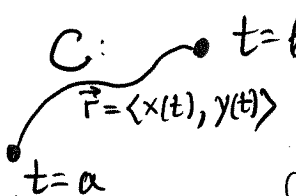
In this section, we'll obtain a generalization of this formula for line integral.
In the remaining sections of Chap. 16, we will further generalize the FTC (for double & triple integrals).

② The Fundamental Thm. for line integrals

Recall: ($\vec{F}$ is conservative) $\Leftrightarrow$ ($\vec{F} = \vec{\nabla} f$ for some $f(x,y,z)$)

Thm: Let C be a contour in 2D or 3D and let $f(x,y,z) \equiv f(\vec{r})$ be such that $\vec{\nabla} f$ is continuous on C. Then:

$$\boxed{\int_C \vec{\nabla} f \bullet d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$



Proof (for 2D; same for 3D):

$$\int_C \vec{\nabla} f \bullet d\vec{r} = \int_C \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bullet \langle dx, dy \rangle$$

$$= \int_C \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bullet \left\langle \frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right\rangle \cdot dt = \int_{t=a}^{t=b} \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt$$

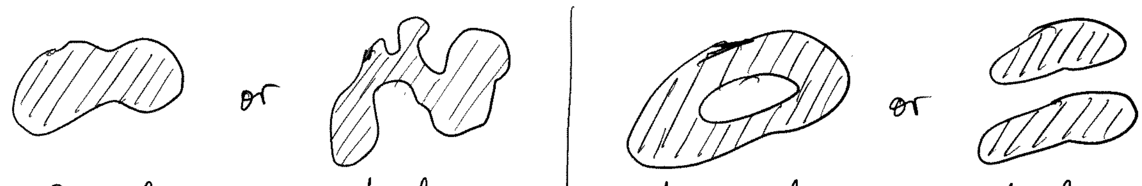
$$\underset{\text{Chain Rule (Sec. 14.5)}}{=} \int_a^b \frac{d f(x(t), y(t))}{dt}\, dt \underset{\text{(FTC)}}{=} f(\vec{r}(t=b)) - f(\vec{r}(t=a)).$$

③ Independence of integration path

We will need the following

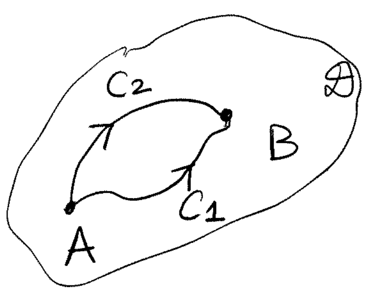
Definition: We call a region $\mathcal{D}$ *simply connected* when:

- it consists of "one piece";
- has no "holes".



Simply connected | Not simply connected

Corollary 1 to the FT Let $\mathcal{D}$ be a simply connected region (in 2D or 3D). Let $\vec{F} = \vec{\nabla} f$, s.t. $\vec{F}$ is continuous everywhere in $\mathcal{D}$. Then for any



two curves, C_1 and C_2, in $\mathcal{D}$ which connect A and B:

$$\boxed{\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}}$$

Proof:

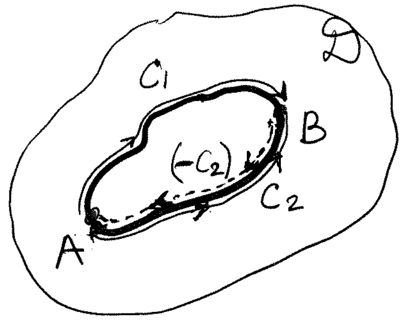
For either C_1 or C_2: $\int_C \vec{F} \bullet d\vec{r} = \int_C \vec{\nabla} f \bullet d\vec{r} \overset{\text{FT}}{=} f(B) - f(A)$, which does not depend on the specifics of the path C.

Note: There remains only one subtle but important question: Why did one have to require that $\vec{F}$ be continuous everywhere in $\mathcal{D}$ instead of just on the curves C_1 and C_2? Preview of answer: # **41** in Sec. 16.3.

Full answer: in Sec. 16.4.

4 Integrals along closed paths

Corollary 2 to the FT Under the same conditions as in Corollary 1,



$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

← This notation, $\oint$, means that contor C **is closed** (i.e., is a loop).

Proof:

Let **C** be the closed contour shown by the **thick line** above. Pick any two points A, B on C. Let:

- C_1 be one part of C, that is directed from A to B;
- C_2 be the other part of C, also directed from A to B;
- $(-C_2)$ be the C_2 traced in the opposite direction (from B to A).

Then:

$$\text{Corollary 1} \Rightarrow \left(\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}\right) \Leftrightarrow \left(\left(\int_{C_1} - \int_{C_2}\right) \vec{F} \bullet d\vec{r} = 0\right)$$

$$\underset{\substack{\text{(topic 3}\\ \text{of Sec. 16.2)}}}{\Longrightarrow} \left(\underbrace{\left(\int_{C_1} + \int_{(-C_2)}\right)}_{=\oint_C,\ \text{since } C = C_1 + (-C_2);\ \text{see figure.}} \vec{F} \bullet d\vec{r} = 0\right) \Leftrightarrow \left(\oint_C \vec{F} \bullet d\vec{r} = 0\right).$$

The results we have obtained so far (in Sec. 16.3 & 16.5, as well as using some material from the book) can be summarized (see next page):

(sec. 16.5; requires $f_{xy}=f_{yx}$, etc.)

$\boxed{1}$ $\vec{F} = \vec{\nabla} f$ $\Longrightarrow$ $\boxed{2}$ $\text{curl}\,\vec{F} = \vec{0}$

$\boxed{1} \Longleftrightarrow \boxed{3}$: (Thm. 16.3.4 (book)) (Corollary 1)

$\boxed{2} \Longleftrightarrow \boxed{4}$: Can be proved using sec. 16.4

$\boxed{3}$ $\int_C \vec{F} \bullet d\vec{r}$ is independent of path C $\Longleftrightarrow$ $\boxed{4}$ $\oint_C \vec{F} \bullet d\vec{r} = 0$ for any closed path C

(Can be easily proved tracing back the steps of Corollary 2's proof.) (Corollary 2)

Thus, statements $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ are all equivalent.

MUST MEMORIZE

⑤ Test for conservativeness

Q1: How can we test of $\vec{F}$ is conservative (i.e., if $\vec{F} = \vec{\nabla} f$ for some f)?

A1: The "$\boxed{2} \Rightarrow \boxed{1}$" in the chart above says: check if $\text{curl}\,\vec{F} = \vec{0}$.

See Ex. 2, 3 in book (and also Ex. 1a below).

Q2: If $\vec{F} = \vec{\nabla} f$, how to find f?

A2: See Ex. 1(b) below.

Ex. 1 (a) (= Ex. 3/book)

Show that $\vec{F} = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2-3y^2}_{Q}, \underbrace{0}_{R} \rangle$ is conservative.

(b) (= Ex. 4(a)/book)

Find f s.t. $\vec{F} = \vec{\nabla} f$.

(c) (= Ex. 4(b)/book)

Find the work done by $\vec{F}$ in moving a particle along path C: $\vec{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$, $0 \le t \le \pi/2$.

in 2D, so no z-component

Sol'n:

(a) Need to show that curl $\vec{F} = \vec{0}$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \vec{i} \cdot \Big(0 - \overbrace{\cancel{\frac{\partial Q}{\partial z}}}^{0}\Big) - \vec{j} \Big(0 - \overbrace{\cancel{\frac{\partial P}{\partial z}}}^{0}\Big) + \vec{k} \cdot \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big);$$

so we need to verify that $Q_x = P_y$.

$$Q_x = (x^2 - 3y^2)_x = 2x - 0$$
$$P_y = (3 + 2xy)_y = 0 + 2x$$

equal. ✓

So, curl $\vec{F} = \vec{0}$ and so $\vec{F} = \vec{\nabla} f$.

(b) We know that $\vec{F} = \langle 3+2xy, x^2-3y^2 \rangle = \langle f_x, f_y \rangle$

(In this calculation we don't need to keep mentioning the z-component of $\vec{F}$, which is 0. See Ex. 5 in book for the case when the z-component is $\neq 0$.)

So $f_x = 3 + 2xy \quad (1)$
$f_y = x^2 - 3y^2 \quad (2)$

$(1) \Rightarrow f = \int (3 + 2xy)\,dx = 3x + x^2 y + K(y)$

Note 1: Here our $\int \ldots dx$ is not along a particular path C, so x & y are independent; so when integrating over x, y is treated as constant (as in Chap. 15).

Note 2: $K(y)$ is some constant that cannot depend on x (since we integrated over x). But it can, and will, depend on y.

We can now substitute the above f into (2):

$$\left(3x + x^2 y + K(y)\right)_y = x^2 - 3y^2 \quad \Rightarrow$$

$$0 + \cancel{x^2} + K_y(y) = \cancel{x^2} - 3y^2 \quad \Rightarrow$$

$$K_y(y) = -3y^2 \Rightarrow K(y) = -y^3 + K_0$$ true constant.

Note 3: **The x-terms in the equation for f_y must always cancel!** If they don't, and you know that $\vec{F}$ is conservative, **then you must have made a mistake**; go back and fix it.

So:

$$f(x,y) = 3x + x^2 y - y^3 + K_0 .$$

(c) Since:

- $\vec{F} = \vec{\nabla} f$ and
- $\vec{F}$ is continuous everywhere,

} conditions of Corollary 1 (topic ③)

we can use Corollary 1 to FT:

$$\int_C \vec{F} \bullet d\vec{r} = f(t=\pi/2) - f(t=0)$$ **regardless of the path C !!**

@ $t=0$ $\quad \vec{r}(0) = \langle e^0 \cdot \cos 0, e^0 \sin 0 \rangle = \langle 1, 0 \rangle$

@ $t=\pi/2$ $\quad \vec{r}(\frac{\pi}{2}) = \langle e^{\pi/2} \cdot \cos\frac{\pi}{2}, e^{\pi/2} \cdot \sin\frac{\pi}{2} \rangle = \langle 0, e^{\pi/2} \rangle$

So

$$\int_C \vec{F} \bullet d\vec{r} = f(\langle 0, e^{\pi/2} \rangle) - f(\langle 1, 0 \rangle) =$$

$$= (0 + 0 \cdot e^{\pi/2} - (e^{\pi/2})^3 + \cancel{K_0}) -$$

$$(3 \cdot 1 + 1^2 \cdot 0 - 0 + \cancel{K_0}) = -e^{3\pi/2} - 3.$$

⑥ Conservation of energy

Any object always has "kinetic energy",

$$E_K = \frac{m|\vec{v}|^2}{2} \quad \left(\begin{array}{l} m = \text{mass}, \\ \vec{v} = \text{velocity} \end{array} \right).$$

If it is also acted upon by a conservative force $\vec{F} = \vec{\nabla} f$, then it also has potential energy:

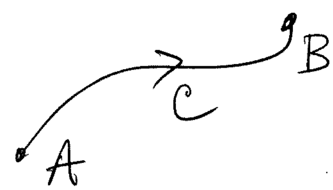
$E_p = (-f)$ (the "−" is the matter of convention).

One can assign this characteristic, the

potential energy, because $E_p = (-f)$ depends only on the current location, $\vec{r} = \langle x, y, z \rangle$, of the object, and not on the path by which the object got to that location.

This follows from the fact that

$$\int_C \vec{F} \bullet d\vec{r} = f(B) - f(A)$$



and does not depend on path C (Corollary 1 to FT).