Sec. 16.4 : Green's Theorem

1 The Main Result

In topic 4 of sec. 16.3 we've seen that if $\vec{F} = \vec{\nabla} f$ (which, as per sec. 16.5 [topic 3, c)], implies $\text{curl}\,\vec{F} = \vec{0}$),

then $\oint_C \vec{F} \bullet d\vec{r} = 0.$

So, what can we say about $\oint_C \vec{F} \bullet d\vec{r}$ when $\text{curl}\,\vec{F} \neq \vec{0}$?

The answer is given by the Thm. below (for 2D).

Before we state it, let us explicitly write $\text{curl}\,\vec{F}$ **when** $\vec{F}$ is in 2D, i.e. $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$

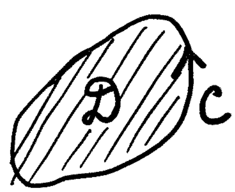
(we did this in Ex. 1(a) for sec. 16.3, but it'll be useful to repeat):

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \vec{i}\Big(\partial_y 0 - \cancelto{0}{\frac{\partial Q(x,y)}{\partial z}}\Big) - \vec{j}\Big(\partial_x 0 - \cancelto{0}{\frac{\partial P(x,y)}{\partial z}}\Big) + \vec{k}\Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)$$

Thus, in 2D, one has:

$$\big(\text{curl}\,\vec{F} \neq \vec{0}\big) \iff \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0\Big).$$

Green's Theorem Let $\mathcal{D}$ be a simply connected (sec. 16.3, topic 3) region where $\vec{F} = \langle P, Q \rangle$ is continuous and has continuous first partial derivatives.



Let C be the boundary of $\mathcal{D}$ (and thus a closed contour) with counter-clockwise orientation. Then:

$$\oint_C \vec{F} \bullet d\vec{r} \equiv \oint_C (P\,dx + Q\,dy) = \iint_{\mathcal{D}} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx\, dy.$$

<u>Note 1:</u> In the notation "$\oint$" we specify (with the arrow, $\uparrow$) the <u>direction</u> in which we trace the contour: $\circlearrowleft$ = counter-clockwise;
$\circlearrowright$ = clockwise.

As we found in sec. 16.2 (topic ③), the choice of the direction of integration affects the sign of $\int \vec{F} \cdot d\vec{r}$ ('line integral of the "work"-type).

<u>Note 2:</u> Green's theorem is **the generalization of the FTC**

$$\int_a^b \underbrace{\frac{dF}{dx}}_{\substack{\text{derivative of } F\\ \text{inside } [a,b]}} dx = \underbrace{F(b) - F(a)}_{\substack{\text{values of } F\\ \text{on the boundary of } [a,b]}}$$

to 2D.

Compare with Green's Thm, written backwards:

$$\iint_{\mathcal{D}} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\substack{\text{partial derivatives of } \vec{F}\\ \text{inside } \mathcal{D}}} dx\,dy = \oint_C \underbrace{(P\,dx + Q\,dy)}_{\substack{\text{values of } \vec{F}\\ \text{on the}\\ \text{boundary of } \mathcal{D}}}$$

<u>Proof of Green's Thm</u>

(done when $\mathcal{D}$ is both Type I & Type II; sec. 15.2)

<u>Want:</u>

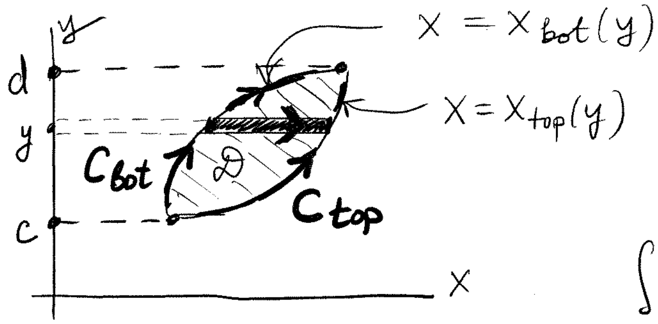
$$\oint_C (P\,dx + Q\,dy) = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\iint_{\mathcal{D}} \left(-\frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx \quad (*) \leftarrow \text{proved in book}$$

$\Rightarrow$

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\, dx\,dy = \oint_C Q\,dy \quad (**) \leftarrow \text{we'll prove below.}$$

<u>Proof of part (**)</u> :



We'll do

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\, dx\, dy$$

as Type II integral:

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} dx\right) dy =$$

$$= \int_c^d \left(\int_{x_{bot}(y)}^{x_{top}(y)} \frac{\partial Q}{\partial x}\, dx\right) dy \underset{\text{FTC!}}{=} \int_c^d \Big(Q(\underbrace{x_{top}(y), y}_{\text{points on } C_{top}}) - Q(\underbrace{x_{bot}(y), y}_{\text{points on } C_{bot}})\Big) dy$$

$$\equiv \int_{C_{top}} Q\, dy - \int_{C_{bot}} Q\, dy \equiv \left(\int_{C_{top}} - \int_{C_{bot}}\right) Q\, dy = \left(\int_{C_{top}} + \int_{-C_{bot}}\right) Q\, dy$$

(path reversal for "work"-type integrals sec. 16.2, topic ③)

$$= \oint_C Q\, dy,$$

because $C = C_{top} + (-C_{bot})$ (see figure).

boundary of $\mathcal{D}$

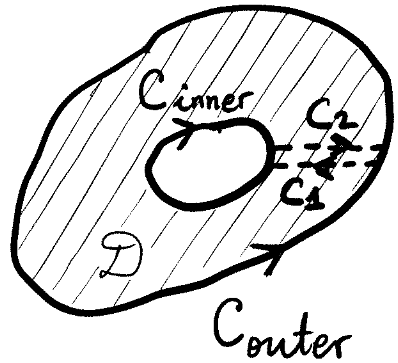
See Ex. 1, 2, 4 for numbers.

<u>Note 1</u>: Green's Thm. can be applied "both ways". I.e., the calculation of the double integral can be reduced to that of the line (i.e., single) integral. However, sometimes it can be easier to compute the double integral than the line integral along a closed curve (see topic ② below). **Green's Theorem is used in very many applications in Engineering and Physics.**

Note 2: One curious (but by far not the most important) application of Green's Theorem is: by measuring coordinates of the boundary of region $\mathcal{D}$, one can compute its area.

See Ex. 3 / book + the text before it, and #25 after sec. 16.4.

② Generalization of Green's Theorem (GT) to not simply-connected regions



Suppose we need to find

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

when $\mathcal{D}$ has a hole.

To apply the GT to this $\mathcal{D}$, one first needs to make it simply connected: "take scissors" and cut $\mathcal{D}$ from the outer boundary to the hole. Let C_1, C_2 be the "banks" of the cut. Then the complete boundary of the new, simply-connected $\mathcal{D}$ is: $C = C_{outer} + C_1 + C_{inner} + C_2$

Note that $C_2 = -C_1$: they trace the same cut, but in opposite directions.

So:

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{\mathcal{D}_{\text{"new"}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\underset{\text{GT}}{=} \left(\int_{C_{outer}} + \int_{C_1} + \int_{C_{inner}} + \int_{C_2}\right)(P\,dx + Q\,dy) =$$

$$= \left(\int_{C_{outer}} + \int_{C_{inner}} + \left[\cancel{\int_{C_1} + \int_{\underset{(-C_1)}{\underset{\shortparallel}{C_2}}}} \right] \right) (P\,dx + Q\,dy)$$

Cancel out because $\int_{-C_1} + \int_{C_1} = 0$ for "work-type" integral

$$= \left(\ointctrclockwise_{C_{outer}} + \ointclockwise_{C_{inner}} \right) (P\,dx + Q\,dy)$$

Thus:

$$\boxed{\iint_{\substack{\mathcal{D} \\ \text{with a hole}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \left(\ointctrclockwise_{C_{outer}} + \ointclockwise_{C_{inner}} \right) (P\,dx + Q\,dy)}$$

Note 1: Note that the orientation of C_{outer} is counter-clockwise (as before), but the orientation of C_{inner} is clockwise (i.e., opposite).

A literate mnemonic way to remember this: When one walks along the boundary in the "correct" direction, the region is always on one's left side.

Note 2: An obvious corollary of the above formula is:

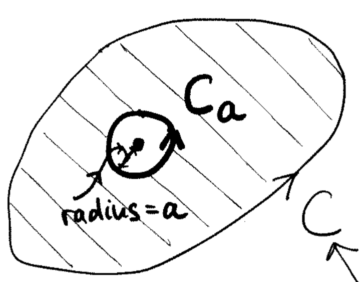
If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ inside a $\mathcal{D}$ with a hole,

then $\boxed{\ointctrclockwise_{C_{outer}} (P\,dx + Q\,dy) = -\ointclockwise_{C_{inner}} (P\,dx + Q\,dy) = \ointctrclockwise_{C_{inner}} (P\,dx + Q\,dy)}$

Note: We did not say "$-C_{inner}$" when going from the middle to the right term because the direction of motion is explicitly indicated:

$$-\ointclockwise_{C} = +\ointctrclockwise_{C}.$$

Ex. 1 (= Ex. 5/book)



Find $\oint_C \vec{F} \bullet d\vec{r}$, where:

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle,$$

and C is any contour enclosing the origin, (0,0).

Sol'n:

1) $\vec{F} \equiv \langle P, Q \rangle$, and one can verify (HW problem #35 for sec. 16.3) that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ except at $x = y = 0$, where **P, Q and their partial derivatives are not continuous.**

Thus, one cannot use the GT inside C. However, if we exclude the origin and define $\mathcal{D}$ to be inside C but outside the circle of some (small?) radius **a**, where $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and both are continuous, then we can use the GT inside this $\mathcal{D}$ "with a hole".

2) According to this GT and Note 2 on p. 32-6,

$$\oint_{\mathbf{C}} \vec{F} \bullet d\vec{r} = \oint_{\mathbf{C_a}} \vec{F} \bullet d\vec{r} \equiv \oint_{C_a} P dx + Q dy.$$

We now follow the steps described in Ex. 4, 5 of sec. 16.2 to compute this $\oint_{Ca}$.

a) Write the eqs. of $(x(t), y(t))$ & (dx, dy) on C_a:

$$x(t) = a\cdot\cos t \qquad 0 \le t \le 2\pi \qquad dx = -a\cdot\sin t\,dt$$
$$y(t) = a\cdot\sin t \qquad\qquad\qquad dy = a\cdot\cos t\,dt$$

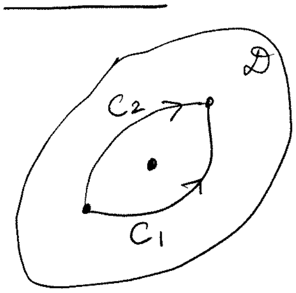
b) substitute these into $\oint_{C_a} P\,dx + Q\,dy$:

$$\oint_{C_a} P\,dx + Q\,dy = \int_{t=0}^{t=2\pi} \left(\frac{(-a\cdot\sin t)}{(a\cos t)^2 + (a\sin t)^2}\cdot(-a\cdot\sin t\cdot dt) + \frac{a\cos t}{a^2}\cdot(a\cos t\,dt) \right)$$

$$= \int_0^{2\pi} \left(\frac{\cancel{a^2}\sin^2 t}{\cancel{a^2}} + \frac{\cancel{a^2}\cos^2 t}{\cancel{a^2}} \right) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi.$$

Note 1: This is a very remarkable result: The work done by this particular force (a very special one, indeed, but having a physical motivation!) around any contour enclosing the origin is 2π. (And if C did not enclose the origin, $\oint_C \vec{F}\cdot d\vec{r}$ would be 0 because $\operatorname{curl}\vec{F} = \vec{0}$ away from the origin.)

Note 2: We can now revisit the "subtle question" posed at the end of topic (3), Sec. 16.3:



we showed that if $\vec{F} = \vec{\nabla} f$ and is continuous everywhere in a simply connected $\mathcal{D}$, then $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

Why did we need the "everywhere"?

In Ex. 1, we had $\vec{F} = \vec{\nabla} f$ for $f = \arctan\frac{y}{x}$ (can verify), which is not continuous at $(x=y=0)$. And if $(0,0)$ lies between C_1 & C_2, then $\int_{C_1} \vec{F}\cdot d\vec{r} \ne \int_{C_2} \vec{F}\cdot d\vec{r}$

3 Vector form of GT

GT: $\iint_{D} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)} dx\,dy = \oint_{C} P\,dx + Q\,dy$

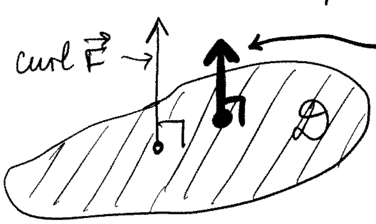
z-component of curl $\vec{F}$, where $\vec{F} = \langle P, Q, 0 \rangle$ (p. 32-1)

How do we "extract" the z-component of a vector?

If $\vec{v} = \vec{i} \cdot a + \vec{j} \cdot b + \vec{k} \cdot c$, then

$$\vec{v} \bullet \vec{k} = \underbrace{(\vec{i} \bullet \vec{k})}_{0} a + \underbrace{(\vec{j} \bullet \vec{k})}_{0} b + \underbrace{(\vec{k} \bullet \vec{k})}_{1} c \quad \text{(sec. 12.3, p. 2-3)}$$

So, the z-component of curl $\vec{F}$ is: $(\text{curl}\,\vec{F}) \bullet \vec{k}$.



$\vec{k}$, = **unit normal vector to the plane containing D**

Then GT can be written as:

$$\boxed{\iint_{D} (\text{curl}\,\vec{F} \bullet \vec{n})\, dA = \oint_{C} \vec{F} \bullet d\vec{r}}$$

unit normal vector to D

We will generalize this in Sec. 16.8.