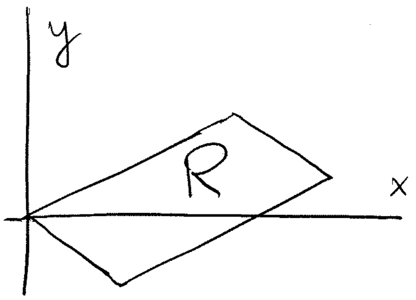
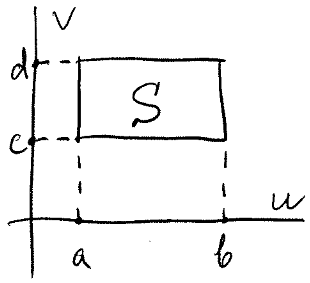
Sec. 16.6: Parametric surfaces

① Motivation and definition

We have seen that sometimes, a change of variables



$$\begin{matrix} u = u(x,y) \\ v = v(x,y) \end{matrix} \quad \xrightarrow{\hspace{2cm}} \quad \xleftarrow{\hspace{2cm}} \quad \begin{matrix} x = x(u,v) \\ y = y(u,v) \end{matrix}$$

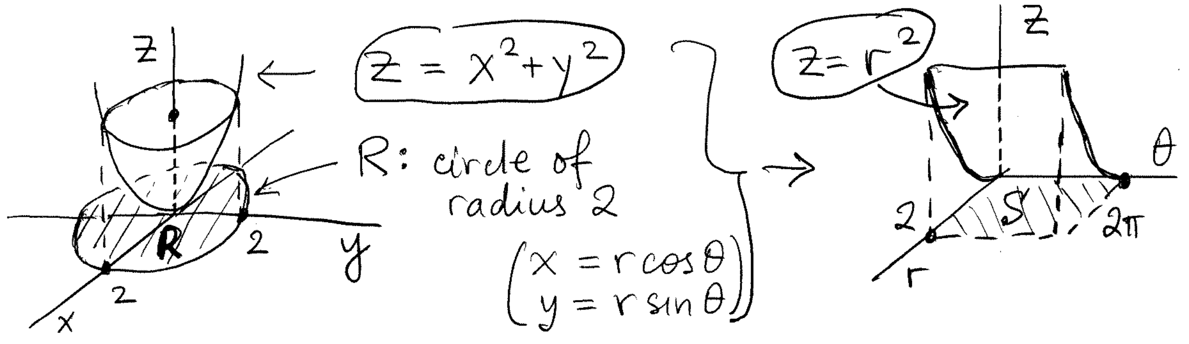


transforms a "complicated" region R in the xy-plane into a simpler region S in the uv-plane.

Then, when we consider a **surface** $z = z(x,y)$, it may be convenient to also define z in terms of the "convenient" coordinates (u,v):

$$z = z\left(x(u,v),\, y(u,v)\right) \equiv f(u,v).$$

This will change the surface from being defined over a "complicated" region R into a (differently shaped) surface over the simpler region S :



So from now on, we will consider **parametric surfaces** defined as:

$$\boxed{x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)}$$

Note 1: This is conceptually analogous to how we defined parametric curves $(x=x(t), y=y(t), z=z(t))$ in Chap. 13. On a curve, one has 1 Degree of Freedom: t. On a surface, one has 2 Degrees of Freedom: u & v.

Note 2: Recall that in this course we always use the notation $\vec{r} \equiv \langle x, y, z \rangle$. Since x, y, z all depend on (u,v), we can now say that a parametric surface is defined by the vector equation $\vec{r} = \vec{r}(u,v)$, which is an equivalent form of $\langle x, y, z\rangle = \langle x(u,v), y(u,v), z(u,v)\rangle$.

In the **Example above**, the **paraboloid** $z = x^2+y^2$, or $\vec{r} = \langle x, y, \underbrace{x^2+y^2}_{z}\rangle$, can be written in **parametric form** as:

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle.$$

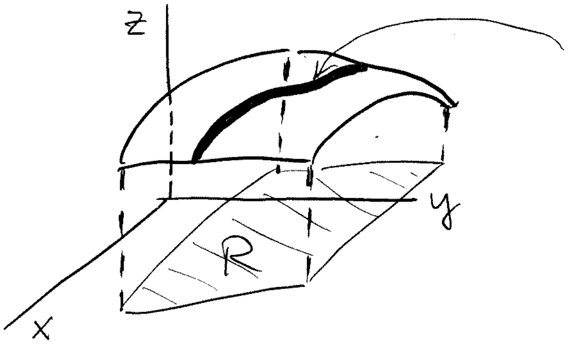
② Tangent plane, normal vector, and surface area of a parametric surface

$\boxed{a}$ **Setup:** Let only one variable change in the parametric surface $\vec{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$.

Say, $\underline{v = v_0}$, u varies; then

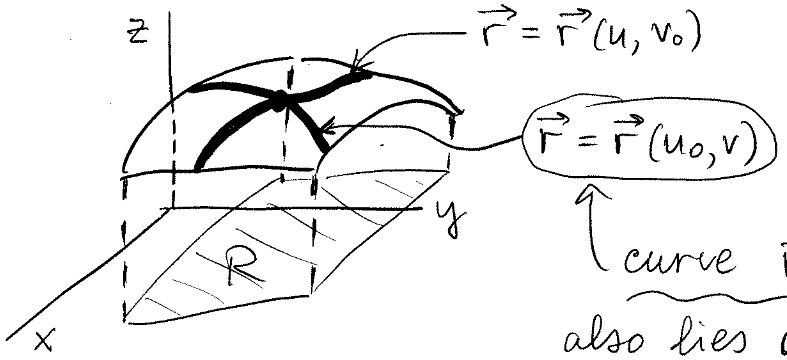
$$\underbrace{\vec{r} = \langle x(u,v_0), y(u,v_0), z(u,v_0) \rangle}$$

For a given v_0, this is exactly the equation of a parametric curve (with $t \to u$) from Chap. 13.



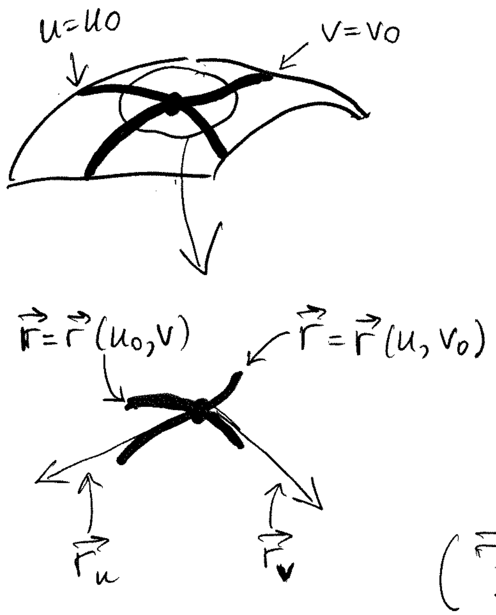
$\vec{r} = \langle x(u,v_0), y(u,v_0), z(u,v_0) \rangle$
$\equiv \vec{r}(u,v_0)$

This curve lies on the parametric surface $\vec{r} = \vec{r}(u,v)$.



Similarly, for any fixed $u = u_0$, the curve $\underset{\sim}{\vec{r} = \vec{r}(u_0,v)}$ also lies on this surface.

$\boxed{b}$ Tangent plane & normal vector



Let us zoom in on the intersection of the curves $\vec{r} = \vec{r}(u,v_0)$ & $\vec{r} = \vec{r}(u_0,v)$ near point (u_0,v_0).

By Sec. 13.2, the **tangent vectors** to these curves are: $\vec{\mathbf{r}}_u$ & $\vec{\mathbf{r}}_v$.

$$\left(\vec{\mathbb{T}}_u = \frac{d\vec{r}(u,v_0)}{du}, \ \vec{\mathbb{T}}_v = \frac{d\vec{r}(u_0,v)}{dv} \right)$$

Since :

- $\vec{r}(u, v_0)$ & $\vec{r}(u_0, v)$ lie on surface $\vec{r}(u,v)$;
- $\vec{r}_u$ is tangent to $\vec{r}(u,v_0)$; $\vec{r}_v$ is tangent to $\vec{r}(u_0,v)$;

$\Rightarrow$ **$\vec{r}_u$ & $\vec{r}_v$ lie in the plane tangent to $\vec{r}(u,v)$ (at point (u_0,v_0)).**

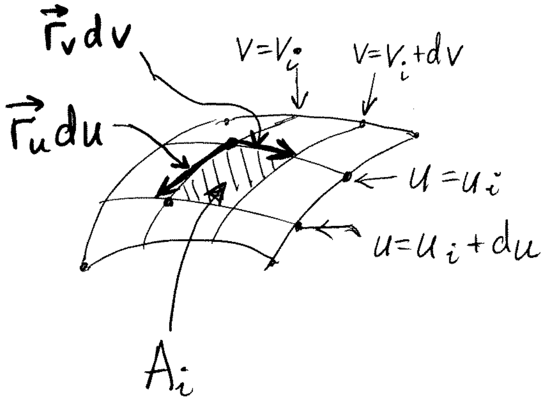
Then, by Sec. 12.4,

$$\boxed{\vec{n} = \vec{r}_u \times \vec{r}_v}$$

is the **normal vector to surface** $\vec{r}=\vec{r}(u,v)$.

Note that the tangent plane at (u_0,v_0) can be found from its two ingredients: the point $\vec{r}(u_0,v_0)$ and the normal vector $\vec{n}$ (see Ex. 9 in the book).

$\boxed{\text{C}}$ Surface area



Suppose we want to find the area of a patch of the surface $\vec{r}=\vec{r}(u,v)$.

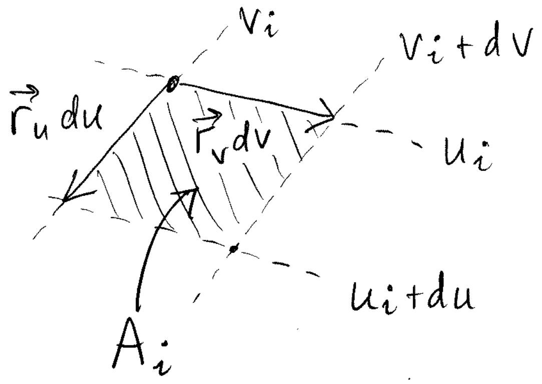
We can draw on it a "curved mesh" by lines $u=u_i (= \text{const})$; $v=v_i (= \text{const})$.

This mesh subdivides the surface into small tiles with areas A_i. Then

$$\underset{\text{area}}{\text{Total}} = \sum_i A_i$$

To find A_i, we recall from Note 4 on p. 26-7

(see also the Figure at bottom of p. 26-10)



the tile is a parallelogram with sides $\vec{r}_u du$, $\vec{r}_v dv$. In the same Note 4, we found:

$$A_i = \left| \vec{r}_u du \times \vec{r}_v dv \right| \Big|_{@(u_i,v_i)},$$

$$\Rightarrow \quad \boxed{A_i = \left| \vec{r}_u \times \vec{r}_v \right| \Big|_{@(u_i,v_i)} \cdot du dv}$$

Then the **total area** of the patch of the surface $\vec{r} = \vec{r}(u,v)$ **"over" the region** S in the uv-plane is:

$$A_{total} = \sum_i A_i \longrightarrow \iint_S \left| \vec{r}_u \times \vec{r}_v \right| du dv .$$

See Ex. 10 in the book and also Ex. 3(b) below. (See also Fig. 14 in the book for a better visualization of a curved mesh subdividing a surface.)

Note: For Cartesian surfaces $z = f(x,y)$, one can take $(u,v) \equiv (x,y)$. Then

$$\vec{r}_u \equiv \vec{r}_x = \langle x, y, f(x,y) \rangle_x = \langle 1, 0, f_x(x,y) \rangle$$
$$\vec{r}_v \equiv \vec{r}_y = \langle x, y, f(x,y) \rangle_y = \langle 0, 1, f_y(x,y) \rangle,$$

and (see the last topic of Sec. 16.6 in the book):

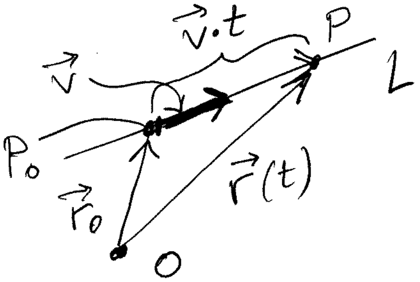
$$A_{total} = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx dy .$$

MUST see Ex. 11 in the book.

③ Parametric eqs. of a plane

Recall from Sec. 12.5-A how the eq. of a line was derived:

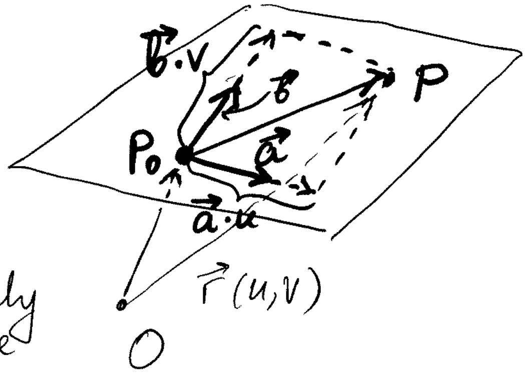
$$\underset{\sim}{\overrightarrow{OP}} = \underset{\substack{\uparrow \\ \text{some pt.} \\ \text{on L}}}{\overrightarrow{OP_0}} + \underset{\substack{\smile \\ \text{going along} \\ \text{L}}}{\overrightarrow{P_0P}}$$



$\vec{r} = \vec{r}_0 + \vec{v}\cdot t$; parameter t describes the 1 Degree of Freedom along the line.

Similarly, when $\vec{P}$ is on a plane "made" by vectors $\vec{a}$ and $\vec{b}$,

$$\boxed{\overrightarrow{OP}} = \underset{\substack{\uparrow \\ \text{some} \\ \text{pt. on plane}}}{\overrightarrow{OP_0}} + \underset{\substack{\smile \\ \text{staying entirely} \\ \text{on the plane}}}{\overrightarrow{P_0P}}$$



In the Addendum to Sec. 12.4 (posted online), we showed that if some two non-parallel vectors $\vec{a}$ & $\vec{b}$ "make" a plane, then any vector $\overrightarrow{P_0P}$ in that plane can be written as:

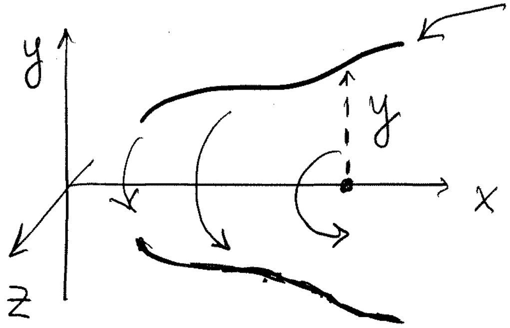
$$\overrightarrow{P_0P} = \vec{a}\cdot u + \vec{b}\cdot v \quad \text{(see the figure above)}$$

for some numbers u, v. Then:

$\boxed{\vec{r}(u,v) = \vec{r}_0 + \vec{a}\cdot u + \vec{b}\cdot v}$ is the param. eq. of a plane. Parameters u & v describe the 2 Degrees of Freedom of motion on a plane.

④ Parametric eqs. of Surfaces of Revolution

- General idea:



Parametric curve

use "u" instead of "t"

$$\begin{cases} x = x(u) \\ y = y(u) \end{cases}$$

When rotating about the **x**-axis:

$$\begin{cases} \boxed{x = x(u)} \\ y = \boxed{y(u)}\cdot \boxed{\cos v} \\ z = \boxed{y(u)}\cdot \boxed{\sin v} \end{cases}$$

← stays the same

radius of rotation

rotation: parametric eqs. of a circle

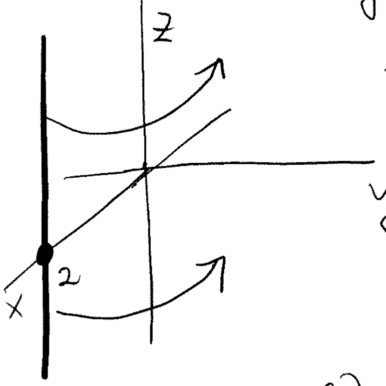
Thus, the parametric eqs. of the Surface of Revolution obtained when a curve is **rotated about the x-axis** are:

$$\begin{cases} x = x(u) \\ y = y(u)\cdot\cos v \\ z = y(u)\cdot\sin v \end{cases}$$

And similarly for surfaces obtained by rotation of curves about the y- and z-axes.

Note: Do NOT follow the "method" on Surfaces of Revolution presented in the book. You will be unable to obtain a new result of Ex. 4 below with the book's "method". You will also be unable to do Lab 7. Thus, using that "method" on a test will earn only 1 point for the effort.

Ex. 1 Find the parametric eqs. of cylinder $x^2 + y^2 = 4$.



Sol'n: 1) This cylinder is obtained by rotating the line $x = 2$ in the xz-plane about the z-axis.

2) Need parametric eqs. of this line:

$$\begin{cases} x = 2 \\ z = u \end{cases}$$ ← any number

3) Rotate about z-axis

$$\begin{cases} x = 2\cdot\cos v \\ y = 2\cdot\sin v \\ z = u \end{cases}$$ ← stays the same

Parametric eqs. of cylinder w/z-axis.

Note: • Rename: $v = ``\theta"$, $u = ``z"$;

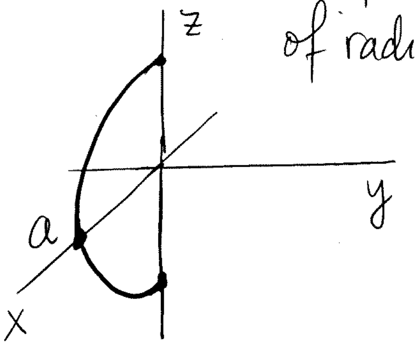
• Allow radius "2" to be any number, "r".

Then these eqs become:

$$\begin{cases} x = ``r"\cdot\cos ``\theta" \\ y = ``r"\sin ``\theta" \\ z = ``z" \end{cases}$$

These are cylindrical coordinates (not surprisingly).

Ex. 2 Find parametric eqs. of a sphere of radius **a** and centered at the origin.



Sol'n: 1) Such a sphere is obtained by rotating a semi-circle (shown) about the z-axis.

(Note that if we rotate a full circle by 2π about the z-axis, we will "cover" the sphere twice.)

2) Parametric eqs. of the semi-circle:

$$\begin{cases} x = a \cdot \sin u \\ z = a \cdot \cos u, \end{cases} \quad 0 \le u \le \pi \leftarrow \text{half-cirle}$$

(We chose the sin & cos in this order for the convenience of a later step.)

3) Rotate about the **z-axis**:

$$\begin{cases} x = (a \cdot \sin u) \cdot \cos v \\ y = (a \cdot \sin u) \cdot \sin v \\ z = a \cdot \cos u \leftarrow \text{stays the same} \end{cases}$$

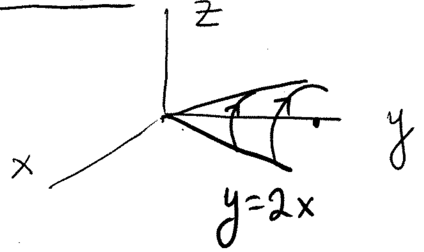
← Parametric eqs. of a sphere of radius **a**.

Rename: $a = "\rho"$, $u = "\phi"$, $"v" = \theta$;
then we recover spherical coord's (not surprisingly).
Incidentally, the Note at the top of this page explains why the range for ϕ is: $0 \le \phi \le \pi$ (Sec. 15.8).

Ex. 3(a) Find param. eqs. of a half-cone

$$y = 2\sqrt{x^2 + z^2}. \quad (y \ge 0)$$

Sol'n:



1) We first recognize that this (half-)cone has the y-axis as its axis.
Thus, it can be obtained by rotating the ray: $y = 2x$, $x \ge 0$, about the y-axis.

2) Parametric eqs. of the ray:

$$\begin{cases} x = u \\ y = 2u, \quad u \geq 0. \end{cases}$$

(Recall – from HW for Sec. 13.1 – that any Cartesian eq. $y = f(x)$ can be written parametrically as: $x = t$, $y = f(t)$.)

3) Rotate about the y-axis:

$$\begin{cases} x = u \cdot \cos v \\ y = 2u \quad \leftarrow \text{ stays the same} \\ z = u \cdot \sin v \end{cases}$$

Parametric eqs. of a cone around the y-axis.

Equivalently, $\vec{r}_{cone}(u, v) = \langle u \cdot \cos v, \ 2u, \ u \cdot \sin v \rangle$.

Ex. 3(b) Find the surface area of the side of this cone for $0 \leq y \leq 3$.

Sol'n: 1) General formula from topic ② $\boxed{c}$:

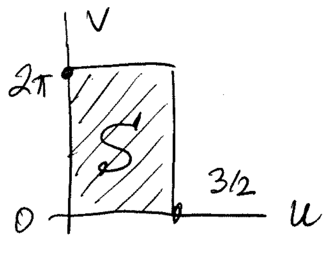
$$A = \iint_S |\vec{r}_u \times \vec{r}_v| \, du dv,$$

where $\vec{r}(u, v)$ is given a few lines above.

2) Determine the limits for S:

$0 \leq y \leq 3 \Rightarrow 0 \leq 2u \leq 3 \Rightarrow \boxed{0 \leq u \leq 3/2}$;

the cone is fully rotated (not by half a turn, etc.), $\Rightarrow \boxed{0 \leq v \leq 2\pi}$.



3) Compute $|\vec{r}_u \times \vec{r}_v|$.

$\vec{r} = \langle u \cdot \cos v, \ 2u, \ u \cdot \sin v \rangle$

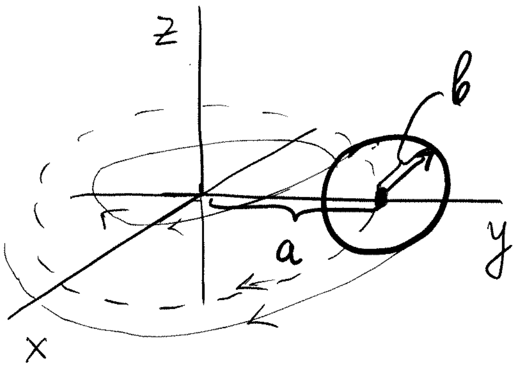
$\vec{r}_u = \langle \cos v, \ 2, \ \sin v \rangle$

$\vec{r}_v = \langle -u \sin v, \ 0, \ u \cdot \cos v \rangle$

$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{2^2 + 1} \cdot u$

4) Sub into formula: $A_{total} = \int_0^{2\pi} \int_0^{3/2} \sqrt{5}\, u \, du dv = \frac{\sqrt{5}}{2} \cdot \left(\frac{3}{2}\right)^2 \cdot 2\pi.$

Ex. 4(a) Find the parametric eqs. of a **torus** (the surface of a donut); see the Figure for #64 after Sec. 16.6.



Sol'n:

1) This torus is obtained by rotating a circle shown on the left around the z-axis.

2) Param. eqs of the circle (radius = b; center @ $(0,a,0)$):

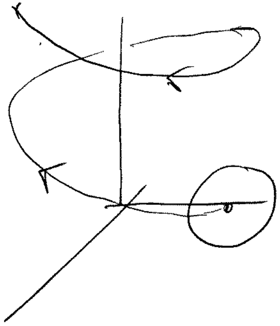
$$\begin{cases} y = a + b\cos u \\ z = b\sin u \end{cases}$$

3) **Rotate about the z-axis:**

$$\boxed{\begin{cases} x = (a + b\cos u)\cdot\cos v \\ y = (a+b\cos u)\cdot\sin v \\ z = b\sin u \quad \leftarrow \text{stays the same} \end{cases}}$$

Param. eqs of a torus, explored in Lab 7.

Ex. 4(b) Find the param. eqs. of a **spiraling tube**:



See Fig. 4, or search online, or think of a "fat" spring (spiral).

1) This tube can be obtained by moving the center of the circle of Ex. 4(a) along a helix (the "must-see" curve from Sec. 13.1, Ex. 4). **This is not a rotation, so the method described above needs a modification.**

2) The "motion" in the xy-plane is still the same as for Ex. 4(a) – rotation (imagine looking at a spiral along its axis: you will see a circle).

However, along the z-axis it is some uniform change; so the **eqs. of the spiraling tube** are

$$\begin{aligned} x &= (a + b\cos u)\cdot\cos v \\ y &= (a + b\cos u)\cdot\sin v \\ z &= b\cdot\sin u + \boxed{c\cdot v} \end{aligned}$$

← same as for the torus

← this new term describes the uniform change

rate of uniform z-change of the center of the circle.

This object will be explored in Lab 7.

⑤ Verifying the parametric equations of a surface

In Ex. 3(a) we derived the parametric eqs. of a cone. Here we will solve an "inverse" problem:

Ex. 5 Verify that the parametric eqs.

$$\begin{cases} x = a\cdot u & 0 \le u \le c \\ y = b\cdot u\cdot\cos v & \frac{\pi}{2} \le v \le \pi \\ z = b\cdot u\cdot\sin v & \end{cases}$$

define a quarter of a half-cone. Find the length and the radius at the base of this cone.

Sol'n: 1) Verify that the eqs. define a cone. The rotation is in the yz-plane, ⇒ cone's axis is along x, ⇒ the general form of its equation must be:

$$x = K\cdot\sqrt{y^2+z^2} \qquad (*)$$

for some $K = \text{const}$. Substitute the given eqs. into the l.h.s. & r.h.s. of (*) and verify (*):

lhs of (*) $= a\cdot u$

$$\text{rhs of } (*) = K\cdot\sqrt{(bu\cdot\cos v)^2+(bu\cdot\sin v)^2}$$

$$= K\cdot\sqrt{(bu)^2\underbrace{\cancel{(\cos^2 v+\sin^2 v)}}_{1}} = K\cdot bu$$

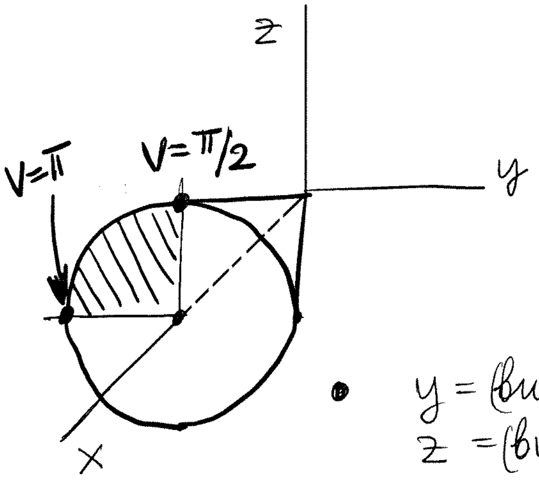
$$\text{lhs} \overset{?}{=} \text{rhs} \Rightarrow a\cdot\cancel{u} = K\cdot b\cdot\cancel{u} \Rightarrow K=\frac{a}{b}.$$

Thus, this cone has the Cartesian equation

$$x = \frac{a}{b}\sqrt{y^2+z^2}.$$

2) Impose the given restrictions on the cone.

SKETCH:



- length:

$$0 \le u \le c$$

$$0 \le \underbrace{a\cdot u} \le ac$$

$$0 \le x \le \boxed{ac}$$

- radius at base:

$$y = b\cdot u_{max} = \boxed{bc}$$

- $y=(bu)\cdot\cos v$, $z=(bu)\cdot\sin v$ $\quad \frac{\pi}{2} \le v \le \pi$.

@ $v=\pi/2$: $(y=0,\ z>0)$ @ $v=\pi$: $(y<0,\ z=0)$

Thus, the dashed part of the cone is what we need.