Sec. 16.7: Surface integrals

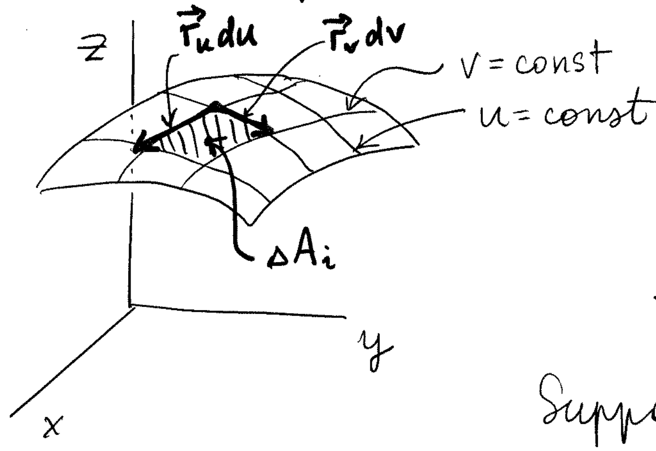
For line integrals (sec. 16.2), we had 2 types:

1: $\int_C f(x,y,z)\,ds$, meaning = mass of a wire C (and similar)

2: $\int_C \vec{F} \cdot d\vec{r}$, meaning = work done by $\vec{F}$ along C.

For surface integrals, we'll also have 2 types.

① Mass of a curved sheet



As in sec. 16.6, consider the param. surface $\vec{r} = \vec{r}(u,v)$, or, equivalently, $\vec{r} = \langle x(u,v), y(u,v), z(u,v) \rangle$.

Suppose this curved sheet has mass surface density $\delta(x,y,z)$. To find the mass of the sheet, we proceed as in sec. 16.6 (topic 2$\boxed{c}$), where we wanted to find the surface area. Namely, we "draw" a mesh of curves (u = const, v = const). This mesh subdivides the sheet into small tiles. Similarly to sec. 16.6 (for the area), we find the mass:

$$M_{total} = \sum_i \Delta M_i \approx \sum_i \delta(x_i, y_i, z_i)\, \Delta A_i \quad \text{sec. 16.6 topic } 2\boxed{c}$$

$$= \sum_i \delta\left(x(u_i,v_i), y(u_i,v_i), z(u_i,v_i)\right) \cdot \boxed{|\vec{r}_u \times \vec{r}_v|\, du\, dv},$$

Thus, total mass of the curved sheet $\vec{r} = \vec{r}(u,v)$:

$$M = \iint_{\mathcal{D}} \delta(x(u,v), y(u,v), z(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, du dv \quad (*)$$

region in (u,v)-plane that corresponds to the particular patch of the surface (see below).

denoted by dS in book; elementary surface area

Note 1: When $\delta = 1$ for all (u,v), the above formula reduces to the formula of the area of the sheet (sec. 16.6).

Note 2: If $(u,v) = (x,y)$ and so $z = f(x,y)$ (the familiar Cartesian surface), then

$$M = \iint_{\mathcal{D}} \delta(x,y,z) \cdot \underbrace{\sqrt{f_x^2 + f_y^2 + 1}}_{} \, dx dy$$

projection of the sheet on the (x,y)-plane

$|\vec{r}_x \times \vec{r}_y|$, see the Note on p. 27-5 (sec. 16.6)

Note 3(a): Note that in the formula $(*)$ above we do not need to "add" the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ "by hand", because we are not changing coordinates from (x,y) to (u,v). Rather, we work with (u,v) from the start.

Note 3(b): The Jacobian, nonetheless, appears in the formula automatically: it is the factor $|\vec{r}_u \times \vec{r}_v|$; see p. 26-7 (sec. 15.9).

See Ex. 1, 2, 3 in book for numeric examples.

Ex. 1 Evaluate the surface integral

$$I = \iint_{S} (y^2 z^2)\, dS,$$

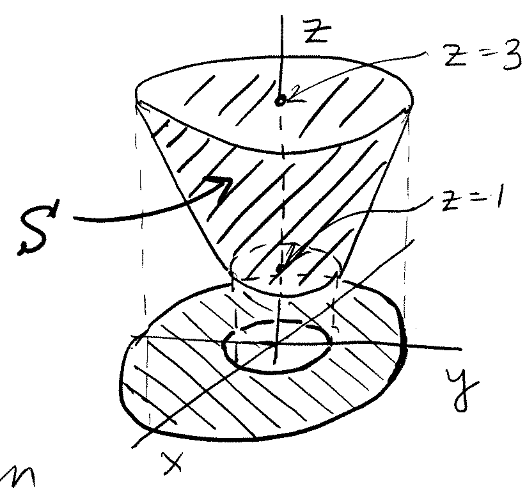
can think of this as the surface mass density

where S is the part of cone $z = 2\sqrt{x^2+y^2}$ that lies between the planes $z=1$ & $z=3$.

The book uses these notations:
S – the curved surface
dS – elementary area on the curved surface
(like previously, in Chap. 15, **dA** meant the elementary area in the xy-plane)

Sol'n:

1) Sketch – to visualize what is being asked.
S = side surface of the cone shown on the right.



2) We found in Ex. 3 of Sec. 16.6 (p. 27-9) that $|\vec{r}_u \times \vec{r}_v|$ and the integration region D look simpler in polar coords:

$$\vec{r} \equiv \langle x, y, z\rangle = \langle r\cos\theta,\ r\sin\theta,\ 2r\rangle.$$

So we use $(u,v) = (r,\theta)$.

3) Find region D (= integration limits) in (r,θ)-plane:

$1 \le z \le 3$ ← given

$1 \le 2r \le 3$ ← since $z = 2r$ on the cone

$$\boxed{1/2 \le r \le 3/2}$$

For θ, we have full rotation, so $\boxed{0 \le \theta \le 2\pi.}$

4) Compute $|\vec{r}_r \times \vec{r}_\theta|$ (similar to Ex. 3/sec. 16.6)

$\vec{r} = \langle r\cos\theta,\ r\sin\theta,\ 2r \rangle$

$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ 2 \rangle$; $\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -2r\cos\theta,\ -2r\sin\theta,\ r \rangle$; $\boxed{|\vec{r}_r \times \vec{r}_\theta| = \sqrt{5}\cdot r}$

5) Substitute and evaluate:

$$\iint_S y^2 z^2 \, dS = \iint_{\mathcal{D}} \underbrace{(r\sin\theta)}_{y}{}^2 \cdot \underbrace{(2r)}_{z}{}^2 \cdot \underbrace{\sqrt{5}\cdot r}_{|\vec{r}_r \times \vec{r}_\theta|} \cdot dr\,d\theta$$

Note again: just $dr\,d\theta$, **not** $r\,dr\,d\theta$

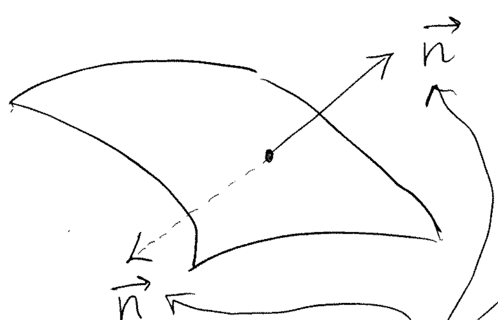
from 3)

$$= \int_0^{2\pi} \sin^2\theta \left(\int_{1/2}^{3/2} \sqrt{5}\cdot r^5 \, dr \right) d\theta = \frac{4}{6}\pi\cdot\sqrt{5}\left(\left(\frac{3}{2}\right)^6 - \left(\frac{1}{2}\right)^6\right)$$

← Answer

half-angle formula (Sec. 7.2, Ex. 3 or Sec. 15.3, Ex. 1/Notes)

② Orientation of a smooth surface



unit normal vector to surface $\vec{r} = \vec{r}(u,v)$

In sec. 16.6 (topic 2B, p. 27-4) we derived:

$$\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Chosen based on where $\vec{n}$ is said to point.

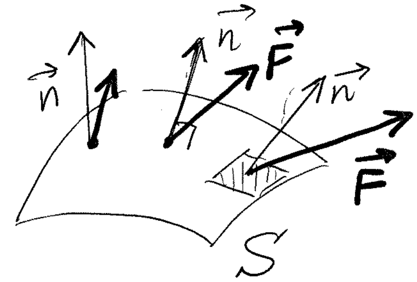
divide by length to make $\vec{n}$ a unit vector

③ Flux of field $\vec{F}$ through surface S

$$\|$$

$$\iint_S (\vec{F} \bullet \vec{n})\, dS$$

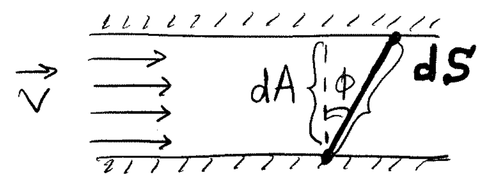
Flux = flow



This is the 2nd type of surface integrals, mentioned on p. 33-1.

Derivation of the above formula for flux:

Consider the uniform flow of a fluid through an elementary area dS:

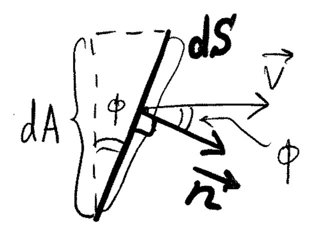


Flow through dS =

flow through dA $(\perp \vec{v})$ =

$$\underbrace{|\vec{v}|}_{\frac{ft}{sec}} \cdot \underbrace{dA}_{ft^2} = |\vec{v}| \cdot (dS \cdot \cos\phi) =$$

$$= \frac{ft^3}{sec}$$



$dA = dS \cdot \cos\phi$

$$= |\vec{v}| \cdot \underbrace{|\vec{n}|}_{1} \cos\phi \cdot dS = (\vec{v} \bullet \vec{n})\, dS$$

Thus, for the flow through a finite sheet S:

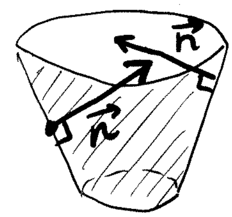
$$\boxed{\begin{pmatrix}\textbf{Flux of } \vec{F} \\ \textbf{through } S\end{pmatrix} = \iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_{\mathcal{D}} \vec{F} \bullet \underbrace{\left(\pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right)}_{\vec{n}} \cdot \underbrace{|\vec{r}_u \times \vec{r}_v|\, du\, dv}_{dS} = \iint_{\mathcal{D}} \vec{F} \bullet (\pm \vec{r}_u \times \vec{r}_v)\, du\, dv}$$

"projection" of S on (u,v)-plane

See Ex. 4,5 in book.

Ex. 2 Let S be the surface of the cone in Ex. 1. Let $\vec{n}$ be directed inside the cone. Find the flux of $\vec{F} = \langle -y, x, 3z^2 \rangle$ through this S.

Sol'n: 1) Write the general formula:



$$\text{Flux} = \iint_{\mathcal{D}} \vec{F} \bullet \left(\pm\, \vec{r}_u \times \vec{r}_v\right) du\, dv$$

2) Specify $u, v, \vec{r}(u,v)$ etc:

In Ex. 1 we used: $(u,v) = (r,\theta)$,

$$\vec{r}_r \times \vec{r}_\theta = \langle -2r\cos\theta, -2r\sin\theta, r \rangle$$

$$\mathcal{D} = \{ 1/2 \le r \le 3/2,\ 0 \le \theta \le 2\pi \}$$

3) **Determine the $\oplus$ or $\ominus$ from the orientation of $\vec{n}$.** ← NEW

- $\vec{n}$ points inside the cone (given) ⇒ **upwards** (see picture) ⇒ z-component of $\vec{n}$ is **positive**.

- $$\vec{n} = \pm \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \pm \frac{\langle -2r\cos\theta, -2r\sin\theta, r \rangle}{\sqrt{5}\, r}$$

z-component of $\vec{n}$; responsible for "up" or "down".

$$(\vec{n})_{z\text{-comp}} = \pm \frac{r}{\sqrt{5}\, r} > 0$$

⇒ must choose $+$.

So need $+\left(\vec{r}_r \times \vec{r}_\theta\right) = \langle -2r\cos\theta, -2r\sin\theta, r \rangle$.

4) Substitute & evaluate:

$$\vec{F} \bullet (\vec{r}_r \times \vec{r}_\theta) = \langle \underbrace{-r\sin\theta}_{-y}, \underbrace{r\cos\theta}_{x}, \underbrace{3\cdot(2r)^2}_{3z^2} \rangle \bullet$$

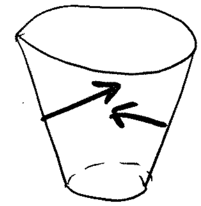
$$\langle -2r\cos\theta, -2r\sin\theta, r \rangle$$

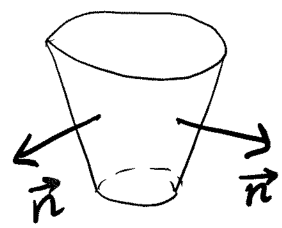
$$= 2r^2\sin\theta\cos\theta - 2r^2\cos\theta\sin\theta + 12r^2\cdot r = 12r^3.$$

$$\text{Flux} = \int_0^{2\pi} \left(\int_{1/2}^{3/2} 12r^3 \cdot dr \right) d\theta = 2\pi \cdot 15.$$

again, note that this is <u>not</u> $r\,dr\,d\theta$.

Interpretation: Since $\vec{n}$ points inside the cone, the above flux is the flow of $\vec{F}$ into the cone. If $\vec{F}$ represents the velocity of some fluid, then its flux measures how much fluid flows <u>into</u> the cone.





On the other hand, if $\vec{n}$ points outside the cone, then the flux measures how much fluid flows <u>out</u> of the cone.