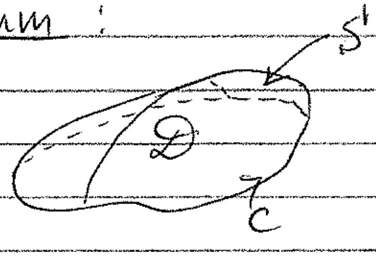
Sec. 16.9. Divergence Thm.
(Gauss Thm.)

(1) Main result.

It extends Green's / Stokes Thm. to 3D.

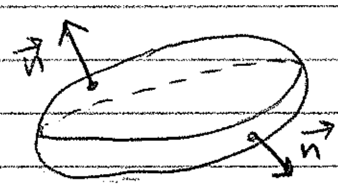
Stokes Thm:



$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$$

Divergence Thm:

"$\bigcirc$" stresses that S is closed.



$$\underbrace{\oiint_S \vec{F} \bullet d\vec{S}}_{\text{Flux through } S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

E = 3D solid

S = its boundary (closed surface)

Thm: Let E be a simply connected solid and S be the boundary (surface) of E, with positive orientation (outside).
Let $\vec{F}$ have continuous partial derivatives in an open region that contains E.
Then:

$$\underbrace{\oiint_S \vec{F} \bullet d\vec{S}}_{\text{Flux through boundary of } E} = \iiint_E \underbrace{\operatorname{div} \vec{F}}_{\text{inside } E} \cdot dV$$

$$\boxed{\text{Make sure to write } \oiint, \text{ not just } \iint. \,!}$$

Note: This is an extension of FTC to 3D; see Sec. 16.4; also the Summary (Sec 16.10) in textbook

Proof

0) Rewrite the statement.

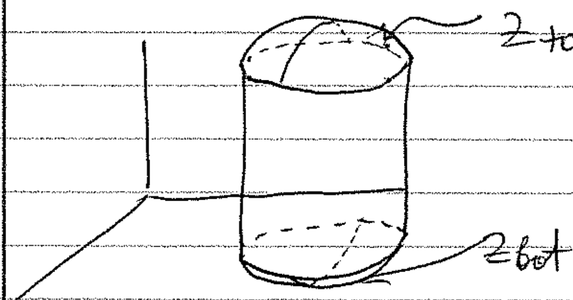
$$\vec{F} = \langle P, Q, R \rangle; \qquad d\vec{S} \equiv \vec{n}\, dS.$$

Then

$$\iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \underline{\frac{\partial R}{\partial z}} \right) dV = \iint_S (P\vec{i} + Q\vec{j} + \underline{R\vec{k}}) \cdot \vec{n}\, dS.$$

Let's prove the R-part; the rest are similar.

1) Suppose E is bounded by top, bottom, and vertical "walls"

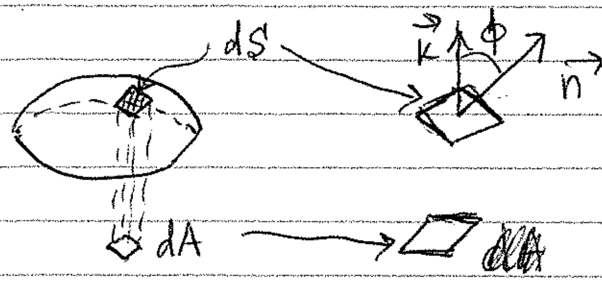


$$\iiint \frac{\partial R}{\partial z}\, dV = \iint_{\mathcal{D}} \left(\int_{z_{bot}}^{z_{top}} \frac{\partial R}{\partial z}\, dz \right) dx\,dy$$

↑ projection of E on xy-plane

$$= \iint_{\mathcal{D}} \left(R(x,y,z_{top}) - R(x,y,z_{bot}) \right) dx\,dy.$$

2)



$$\boxed{dA = dS \cdot \cos\phi}$$

(see Sec. 16.7)

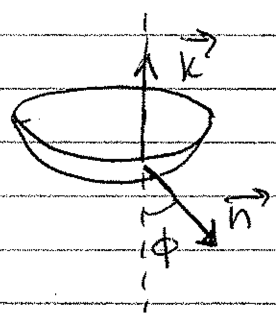
Then at the top surface:

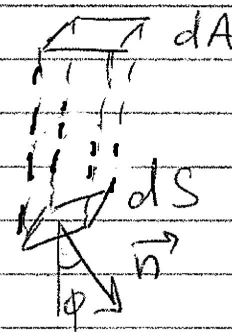
$$\iint_{D} R(x,y,z_{top})\,dA = \iint_{D} R(x,y,z_{top})\cdot\cos\phi\cdot dS$$

$$= \iint_{S_{top}} R(x,y,z)\ \vec{k}\cdot\vec{n}\ dS.$$

on S_{top}, $z = z_{top}$!

3) At the bottom surface:





Now, again, $dA = dS\cdot\cos\phi$,

but

$\cos\phi = -\,\vec{k}\cdot\vec{n}$,

because $\vec{k}\cdot\vec{n} < 0$.

So $$\iint_{D} R(x,y,z_{bot})\,dA = \iint_{S_{bot}} R(x,y,z)\,(-\vec{k}\cdot\vec{n})\,dS$$

4) Thus,

$$\iiint_{E} \frac{\partial R}{\partial z}\,dV = \iint_{D} \Big(R(x,y,z_{top}) - R(x,y,z_{bot})\Big)\,dA$$

$$= \iint_{S_{top}} R\,\vec{k}\cdot\vec{n}\,dS - \iint_{S_{bot}} R\cdot(-\vec{k}\cdot\vec{n})\,dS$$

$$= \iint_{(S_{top}+S_{bot})} R\cdot\vec{k}\cdot\vec{n}\,dS$$ (on the side walls, $\vec{n}\perp\vec{k}$ $\Rightarrow \vec{k}\cdot\vec{n} = 0$)

$\Rightarrow$

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \oiint_S R\, \vec{k} \cdot \vec{n}\, dS.$$

S = full boundary of E

Significance of Div. theorem:

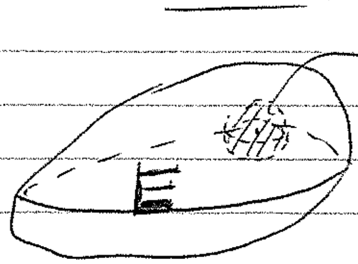
Can compute the flux of $\vec{F}$ through closed surface S by evaluating the

$\iiint_E \operatorname{div}\vec{F}\, dV$, where E is bounded by S.

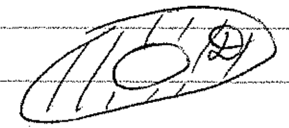
(often, $\operatorname{div}\vec{F}$ has a simpler form than $\vec{F}\cdot\vec{n}$.)

See Ex. 1, 2 in book.

② Extension of Div. Thm. when E has holes.



Similarly to a derivation in Sec. 16.4 where $\mathcal{D}$ had a hole:

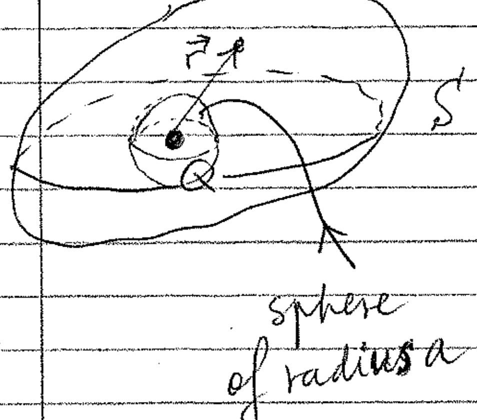


here we have:

$$\iiint_E \operatorname{div}\vec{F}\, dV = \left(\oiint_{S_{outside}} + \oiint_{S_{cavity}} \right) \vec{F}\cdot\vec{n}\, dS,$$

where $\vec{n}$ points outside on $S_{outside}$;
$\vec{n}$ points inside on S_{inside}.

Ex. 1 In electrostatics, the following problem is important.



Find the flux of electric field $\vec{E}$ through *any* surface S enclosing a point charge Q.

Sol'n: 1) $\vec{E} = \dfrac{Q\vec{r}}{|\vec{r}|^3}$, $\vec{r} = \langle x, y, z\rangle$.

2) $\vec{E}$ (and $\operatorname{div}\vec{E}$) are not continuous at $\vec{r}=0$, $\Rightarrow$ cannot apply the Div. Thm. to the region enclosed by S.

But can apply it to a region with a hole around $\vec{r}=0$!

So: $$\left(\oiint_S + \oiint_{S_{cavity}}\right) \vec{E}\cdot d\vec{S} = \iiint_{E \text{ with a cavity}} \operatorname{div}\vec{E}\, dV$$

3) Verify by direct calculation (tedious!) that

$$\operatorname{div}\vec{E} = Q \operatorname{div} \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}^{\,3}} = 0 \quad (\text{for } \vec{r} \neq 0)$$

Then $$\left(\oiint_S + \oiint_{S'} \vec{E}\cdot d\vec{S}\right) = \iiint 0\, dV = 0.$$

So $\oiint_{S} \vec{E} \bullet d\vec{S} = -\oiint_{S_{cavity}} \vec{E} \bullet d\vec{S} = \oplus \oiint_{\underbrace{\text{"}-S_{cavity}\text{"}}} \vec{E} \bullet d\vec{S}$

surface of the cavity with $\vec{n}$ pointing outside

4) On S', $|\vec{r}| = a$, $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$. Then

$$\oiint_{\text{"}-S_{cavity}\text{"}} \vec{E} \bullet d\vec{S} = \oiint_{\text{"}-S'_{cavity}\text{"}} \frac{Q\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|} dS$$

$$= Q \oiint_{\text{"}-S'_{cav.}\text{"}} \frac{\vec{r} \bullet \vec{r}}{|\vec{r}|^4} dS = Q \oiint_{\text{"}-S'_{cav.}\text{"}} \frac{1}{a^2} \cdot dS$$

$$= \frac{Q}{a^2} \cdot \oiint_{\underbrace{\text{"}-S_{cavity}\text{"}}_{\text{surface of a sphere of radius } a}} dS' = \frac{Q}{\cancel{a^2}} \cdot 4\pi \cancel{a^2} = 4\pi Q.$$

Thus:

$$\boxed{\oiint_{S'} \frac{Q\vec{r}}{|\vec{r}|^3} \bullet d\vec{S} = 4\pi Q \quad \text{for any surface surrounding } Q}$$

Gauss Law

closed