Review of Cross-Product (Sec. 12.4) (3-1)

① Definition

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

9th Ed: See formulas $\boxed{6}$, $\boxed{7}$ on p. 856

Note: cross-product is a vector !!!

② Main properties of cross-product

1) $\vec{v} \times \vec{u} = - \vec{u} \times \vec{v}$

2) Corollary of 1):

$\vec{u} \times \vec{u} = \vec{0}$. More generally:

$$\vec{u} \times (\underbrace{k}_{\text{any scalar}}\vec{u}) = k \cdot \underbrace{(\vec{u} \times \vec{u})}_{\vec{0}} = \vec{0}.$$

But $k\vec{u} \parallel \vec{u}$ (Sec. 12.2). Thus:

$$\boxed{(\vec{u} \times \vec{v} = \vec{0}) \Leftrightarrow (\vec{u} \parallel \vec{v})}$$

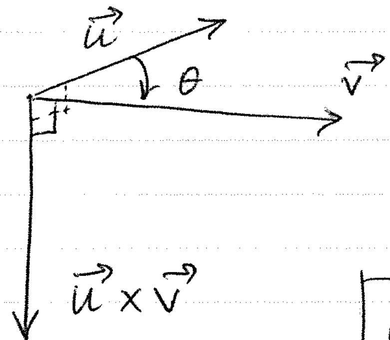
think of this as $k\vec{u}$.

- This formula can be used to check if $\vec{u} \parallel \vec{v}$ when we don't "see" components of $\vec{u}$ & $\vec{v}$ as numbers.
- (when we see them as numbers, we just do an inspection if $\vec{v} = k \cdot \vec{u}$ for some k).

→ Other properties: **must** see p. 859 (e-book). (9th Ed.)

3 Geometric meaning & purpose of cross-product

direction: "right-hand rule"

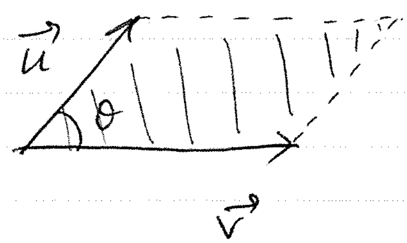


$$\boxed{\begin{aligned} (\vec{u} \times \vec{v}) &\perp \vec{u} \\ (\vec{u} \times \vec{v}) &\perp \vec{v} \end{aligned}}$$

Thus,

$$\boxed{(\vec{u} \times \vec{v}) \perp \left(\text{plane made by } \vec{u} \,\&\, \vec{v}\right)}$$

The main use of cross-product:
it is a vector that is $\perp$ to two given vectors.

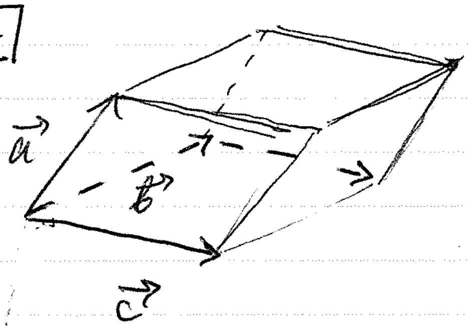


$$\underbrace{|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta}$$

(magnitude of $\vec{u} \times \vec{v}$)
=
(area of parallelogram made by $\vec{u}$ & $\vec{v}$.)

4 Triple product

$\boxed{a}$



$$V = |\vec{a} \bullet \underbrace{(\vec{b} \times \vec{c})}|$$

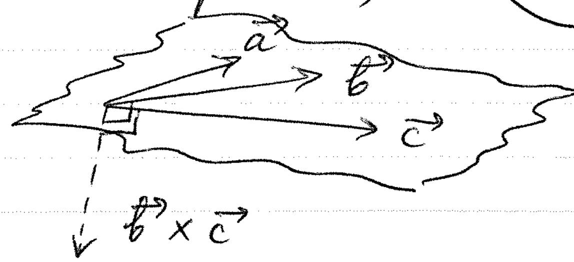
V: volume of slanted box made by $\vec{a}, \vec{b}, \vec{c}$

$(\vec{b} \times \vec{c})$: in any permutation

Formula: $\vec{a} \bullet (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

3-3

b If $\vec{a}, \vec{b}, \vec{c}$ lie in same plane, then $V = 0$:



(We can also see that in this case $\vec{a} \perp (\vec{b} \times \vec{c})$,

$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c})$ must $= 0$ ← see Sec. 12.3.)

Thus:

$$\boxed{\left(\vec{a}, \vec{b}, \vec{c} \text{ lie in same plane}\right) \Longleftrightarrow \left(\vec{a} \cdot (\vec{b} \times \vec{c}) = 0\right)}$$

See Ex. 5 in book for numbers.

C In Lecture 1, topic 2c we showed:

$$\left(\begin{matrix}\vec{a}, \vec{b}, \vec{c} \text{ lie in same plane} \\ (\text{and } \vec{a} \nparallel \vec{b})\end{matrix}\right) \Longrightarrow \left(\begin{matrix}\vec{c} = s\vec{a} + t \cdot \vec{b} \\ \text{for some scalars } s, t\end{matrix}\right)$$

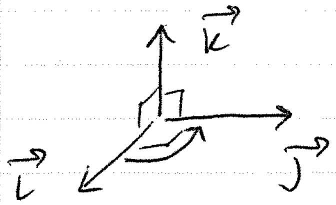
Now we can show $\Longleftarrow$.

Proof: Let $\vec{c} = s\vec{a} + t\vec{b}$.
To test if $\vec{c}$ is in same plane with $\vec{a}$ & $\vec{b}$, compute triple prod:

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = (s\vec{a} + t\vec{b}) \cdot (\vec{a} \times \vec{b})$$
$$= s \cdot \vec{a} \cdot \underbrace{(\vec{a} \times \vec{b})}_{\perp \vec{a}} + t \cdot \vec{b} \cdot \underbrace{(\vec{a} \times \vec{b})}_{\perp \vec{b}}$$
$$= s \cdot 0 + t \cdot 0 = 0. \checkmark$$

So, $\vec{c}$ is in same plane as $\vec{a}, \vec{b}$. q.e.d.

⑤ Cross-product of $\vec{i}, \vec{j}, \vec{k}$. (3-4)



$$|\vec{i} \times \vec{j}| = \underbrace{|\vec{i}|}_{1} \cdot \underbrace{|\vec{j}|}_{1} \underbrace{\sin\theta}_{\sin 90^\circ = 1} = 1.$$

Direction: $(\vec{i} \times \vec{j}) \perp$ xy-plane, points up.

Thus,

$$\boxed{\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{aligned}}$$ (so $\vec{i} \times \vec{k} = -\vec{j}$)

Also, $\vec{i} \times \vec{i} = \vec{0} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k}$.
(see topic ② above).

HW #3 TF (p. 842) ④, 6, 7, 9, 13, 14, 20, 21.

Sec. 12.4:
13, 3 – meaning, def.
11 – basic properties
16 – r.h.-rule
29, 19 – vector $\perp$ to two given ones
37, 38 – coplanar