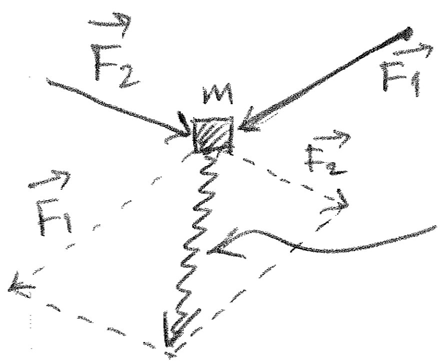
## ⑥ Applications of cross-product to Mechanics.

### [a] The 2nd law of Newton for the rotational motion. Torque.

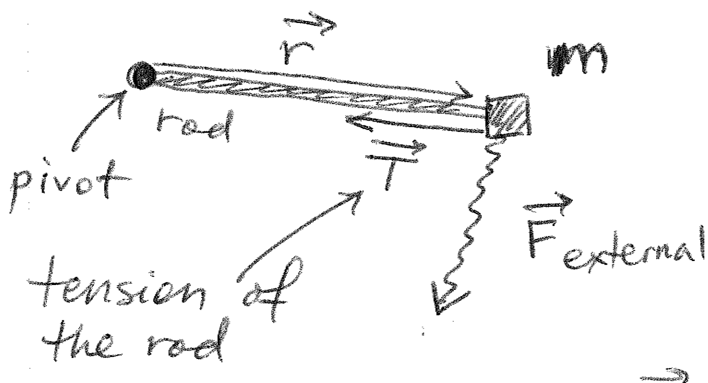
Recall the "regular" 2nd Law of Newton:

$$m\vec{a} = \underbrace{\Sigma \vec{F}}_{}$$ sum of all forces.




$\vec{F_1} + \vec{F_2} = \Sigma_i \vec{F}$.
In what follows we'll drop the "$\Sigma_i$". So:

$$m\vec{a} = \vec{F} \leftarrow$$ sum of all external forces.




Now consider a mass rotating about a pivot.

2nd Law becomes:

$$m\vec{a} = \vec{F}_{ext} + \vec{T} \leftarrow$$ tension of rod.

<u>Problem</u>: We do <u>not</u> know $\vec{T}$! (It can be found, but it takes effort.)

Q: How can we exclude the need to know this $\vec{T}$?

Note: $\vec{T} \parallel \vec{r}$.

So, answer: cross-multiply by $\vec{r}$!

$$m(\vec{r}\times\vec{a}) = \vec{r}\times\vec{F}_{ext} + \underbrace{\cancel{\vec{r}\times\underset{\parallel \vec{r}}{\vec{T}}}}_{\vec{0}}$$

gone!

So, we arrive at the 2nd Law of Newton for rotational motion:

$$\boxed{m(\vec{r}\times\vec{a}) = \underbrace{\vec{r}\times\vec{F}_{ext}}_{\text{torque, or moment}}}$$

torque, or moment (book uses $\vec{\tau}$; engineering texts use $\vec{M}$).

6 Another form of the same. Angular momentum.

Recall that "regular" 2nd Law of N. can we written as

$$\frac{d(m\vec{v})}{dt} = \vec{F}$$

(because $\vec{a} = \frac{d\vec{v}}{dt}$, $m=const$).

$m\vec{v}$ is called "momentum".

We'll establish a similar-looking ~~form~~ form for the rotational motion.

$$m(\vec{r} \times \vec{a}) = \vec{M}.$$

We'll only transform the l.h.s., starting with $\vec{a} = \frac{d\vec{v}}{dt}$:

$$m\left(\vec{r} \times \frac{d\vec{v}}{dt}\right) \underset{\uparrow}{=} m\left[\frac{d}{dt}(\vec{r}\times\vec{v}) - \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} \times \vec{v}\right]$$

product rule:
$f \cdot g' = (fg)' - f'g$

$$= m\left[\frac{d}{dt}(\vec{r}\times\vec{v}) - \cancelto{\vec{0}}{\vec{v}\times\vec{v}}\right] = \frac{d}{dt}\underbrace{(m\vec{r}\times\vec{v})}_{\text{angular momentum}}$$

Thus, 2nd Law of N. for rotational motion is:

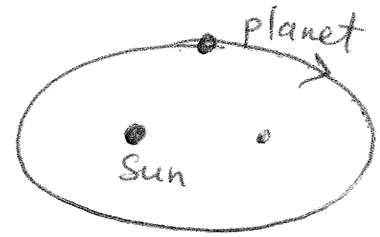
$$\boxed{\frac{d}{dt}(m\vec{r}\times\vec{v}) = \vec{M}}$$

## [c] Kepler's laws of planetary motion

In 1609 Johannes Kepler, the director of the Prague observatory (Chekh Kingdom) published 2 empirical laws, which he deduced from the observational data collected (~~without~~ before the invention of the telescope!) by his predecessor, Tycho Brahe.

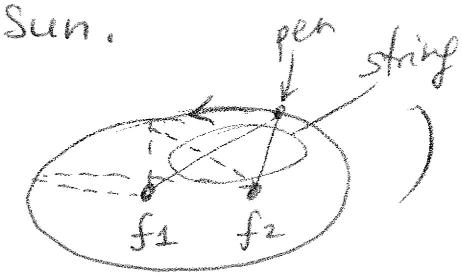
These laws were (stated in their modern form):

**1 Law of Kepler**:




the planets' orbits are ellipses, in one of whose foci is the Sun.

("Focus" of ellipse:

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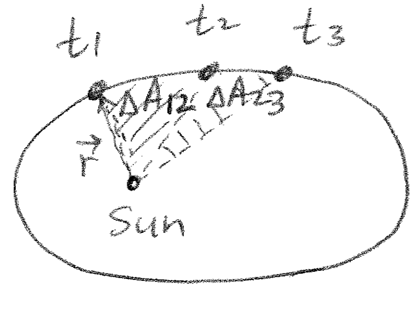
**2nd Law of Kepler**:




The line joining the Sun to a planet sweeps out <u>equal areas</u> in <u>equal times</u>;

if $t_2 - t_1 = t_3 - t_2$ $(= \Delta t)$, then

$$\Delta A_{12} = \Delta A_{23}$$

<u>Note!</u> These laws improve upon those of Copernicus, which were:

<u>1st Copernican law</u>: Planets' orbits are circles with Sun at the center;

(early 16<u>th</u> century)

2nd Copernican law: Planets move with uniform speeds.

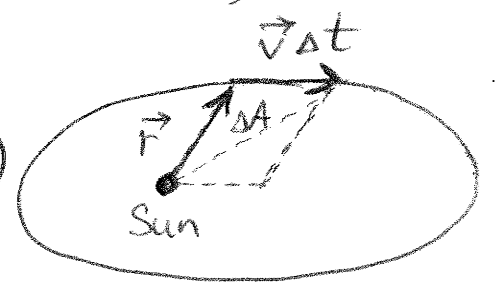
Below we'll show that Kepler's 2nd Law:

(i) implies the conservation of the angular momentum

(ii) and then this conservation implies a certain property of $\vec{F}_{gravity}$.

Note: These implications could be made only after Newton stated "his" laws (2nd Law of N.) at the end of the 17 th century (i.e. more than 70 years after Kepler stated his laws).

(i) $\Delta A = \frac{1}{2}\left(\begin{matrix}\text{area of}\\ \text{parallelogram}\end{matrix}\right)$




$= \frac{1}{2}\,|\vec{r} \times \vec{v}\Delta t|$

$= \frac{1}{2}\Delta t \cdot |\vec{r} \times \vec{v}|, \quad \Rightarrow$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\,|\vec{r} \times \vec{v}|.$$

2nd Law of K. $\Rightarrow \frac{\Delta A}{\Delta t} = \text{const}$,

then $|\vec{r} \times \vec{v}| = \text{const}$. Then:

- $\underbrace{m \, |\vec{r} \times \vec{v}|}_{\text{length of } \overrightarrow{\text{ang. momentum}}} = \text{const}$

- $\vec{r} \times \vec{v} \perp$ the plane of the ellipse,
  $\Rightarrow$ the direction of $\overrightarrow{\text{ang. momentum}}$ is const.

thus, vector $m(\vec{r} \times \vec{v}) = \overrightarrow{\text{const}}$, or

$$\frac{d}{dt}(m\,\vec{r} \times \vec{v}) = \vec{0}.$$

(ii) Recall 2nd Law of N. for rotational motion in the form:

$$\underbrace{\frac{d}{dt}(m\vec{r} \times \vec{v})}_{=\vec{0}} = \underbrace{\vec{r} \times \vec{F}_{gravity}}_{=\vec{0}}$$

the only force on planet

$= \vec{0}$, $\Rightarrow$

$$\vec{r} \times \vec{F}_{gravity} = \vec{0} \Rightarrow \boxed{\vec{F}_{gravity} \parallel \vec{r}.}$$

Thus, the gravity force is <u>central</u> (directed along $\vec{r}$).