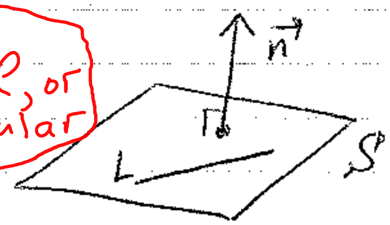
Review of Planes in 3D (sec. 12.5) (5-1)

(1) Equation of a plane. "Ingredients" of a plane

- Recall an intuitive fact:

  Means: orthogonal, or perpendicular

  If $\vec{n}$ is the normal vector to plane S, then $\vec{n} \perp$ to any line L in S.



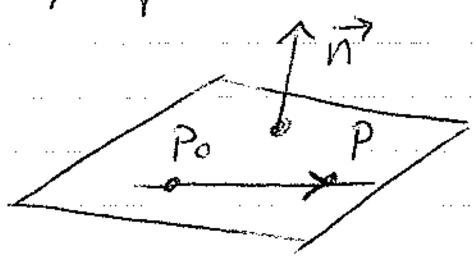
Problem: **Given:**

- point $P_0(x_0, y_0, z_0)$ in S;
- $\vec{n} = \langle a, b, c \rangle$ normal to S.

"Ingredients" of plane !!!

Find: Eq. of S.

Solution:



Take any pt. P in S.
Then line $P_0P \perp \vec{n}$,
or $\overrightarrow{P_0P} \perp \vec{n}$:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \Rightarrow \langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$\Rightarrow \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0} \quad (\star)$$

or

$$\boxed{ax + by + cz = d} \quad (\star\star)$$

Thus: $\boxed{\left(\text{given the eq. of plane, } (\star) \text{ or } (\star\star)\right) \Longleftrightarrow \left(\text{given two "ingredients", } P_0 \ \& \ \vec{n}\right)}$

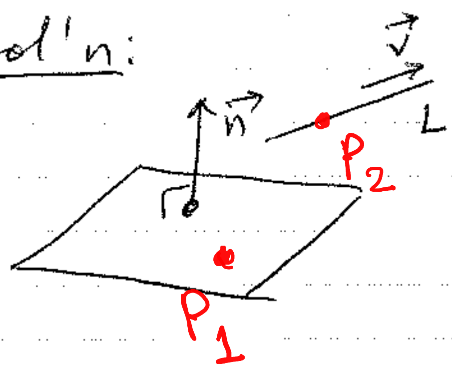
Note: If you are "given" (★);
then coord's of one point in that plane are "read off" automatically: (x_0, y_0, z_0).

If you are given (★★), then you can determine a point in S, by, e.g.:
Set $x = 0 = y$; then
$$a \cdot 0 + b \cdot 0 + c \cdot z = d \Rightarrow z = d/c,$$
$$\Rightarrow P = (0, 0, d/c).$$
Obviously, this is one out of ∞ many points.

Ex. 1 Determine if the given line & plane are parallel:

$S: 2x - y + z = 3$; L: $x = 3t$, $y = 5t$, $z = -t + 4$

Sol'n:



0) SKETCH, including all the ingredients.

1) $L \parallel S$ iff $\vec{v} \perp \vec{n}$.

Thus, we need to read off $\vec{v}$ and $\vec{n}$ from the given eqs.

2) $\vec{v} = \langle 3, 5, -1 \rangle$; $\vec{n} = \langle 2, -1, 1 \rangle$.

3) It remains to check:
$$\vec{v} \circ \vec{n} = \langle 3, 5, -1 \rangle \circ \langle 2, -1, 1 \rangle \overset{\checkmark}{=} 0$$
Yes, $L \parallel S$.

2 Why do we need 1 eq. for a plane?

0 eqs $\longrightarrow$ $\mathbb{R}^3$ (no constraints on x, y, z).

1 eq. $\longrightarrow$ one variable is constrained, $\Rightarrow$ two variables remain free $\Rightarrow$ $\mathbb{R}^2$.

$$\boxed{3 - 1 = 2}$$

degrees of freedom in $\mathbb{R}^3$

constrain 1 variable

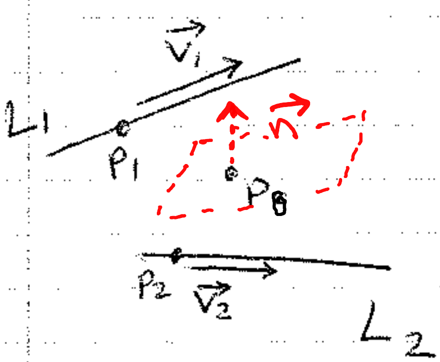
2 variables define a plane.

3 Some other ways to define a plane

Main point: reduce each new problem to the standard form/ingredients considered in topic 1.

(Same idea as in topic 4 of Lec. 12.5-A.)

a Plane through a pt. and // to two given lines



(Dashed lines: things that you need to find.)

Sol'n: 1) List given ingredients in your sketch.

2) For plane S, we have

- point P_0. ✓
- ⊖ still need $\vec{n}$.

5-4

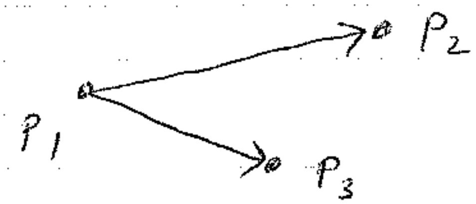
by Ex. 1

3) We want $S \parallel L_1 \Rightarrow \vec{n} \perp \vec{v_1}$
$S \parallel L_2 \Rightarrow \vec{n} \perp \vec{v_2}$.

$$\therefore \left(\vec{n} \perp \vec{v_1} \;\&\; \vec{v_2}\right) \Rightarrow \left(\vec{n} = \vec{v_1} \times \vec{v_2}\right).$$

Thus, we have both ingredients for S, $\Rightarrow$ know its equation.

$\boxed{b}$ Plane through three points



Given: P_1, P_2, P_3

Find: S, i.e.:

$P_1 \longrightarrow$ • a pt. in S;

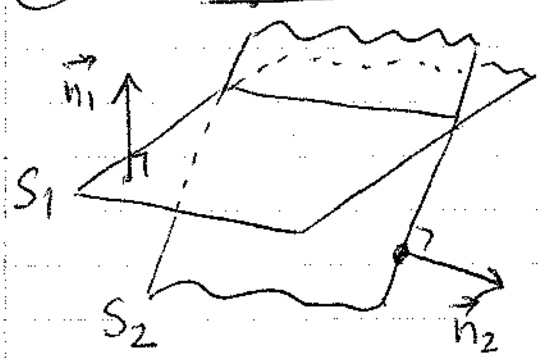
• $\vec{n}$.

S contains $\vec{P_1P_2}$, $\vec{P_1P_3}$
$\Rightarrow$ is $\parallel$ to these vectors.

Thus by $\boxed{a}$, $\vec{n} = \vec{P_1P_2} \times \vec{P_1P_3}$.

See Ex. 5 in book for numbers.

④ Angle between planes $= \underbrace{\angle(\vec{n_1}, \vec{n_2})}$.



and by convention it is taken to be $\leq 90°$
(if it is $> 90°$, simply change the direction of $\vec{n_2}$ to opposite).

See topic ① in Lec. ~~12~~ 3 (dot product) and Ex. 7(a) in book. (Sec. 12.5)

⑤ Intersection between line & plane

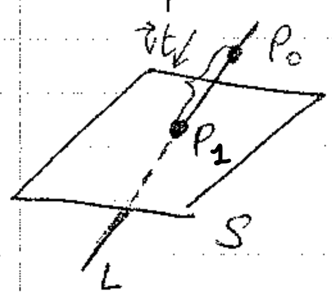
Ex. 2 (extends Ex. 6 in book)
Find the point where line $L: \frac{x-2}{3} = \frac{y}{-4} = \frac{z-5}{1}$ intersects plane $S: 4x+5y-2z=18$.

Sol'n: For this type of problems it is most convenient to use parametric eqs. of L:

$$L: \ x = 2+3t, \ y = 0-4t, \ z = 5+t. \quad (*)$$

Recall the meaning of parameter t:

- it indicates the location of a point on L (e.g., as at different times we find the point at different locations).



So, we just need to find the "time" t at which the point on the line will cross the plane.

For this, substitute $(*)$ into the eq. of S:

$$4\cdot\underbrace{(2+3t)}_{x} + 5\cdot\underbrace{(0-4t)}_{y} - 2\underbrace{(5+t)}_{z} = 18$$

Solve this eq. for t: $t=-2$.

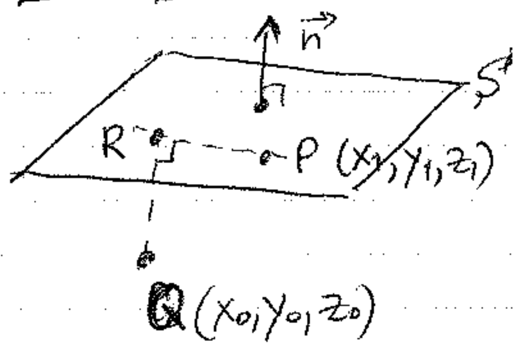
Substitute back into $(*)$:

$$x_1 = 2+3\cdot(-2) = \boxed{-4}$$
$$y_1 = -4\cdot(-2) = \boxed{8}$$
$$z_1 = 5+(-2) = \boxed{3}$$

coord's of the intersection point P_1

6 Distance problems involving planes

a Distance between a pt. & plane



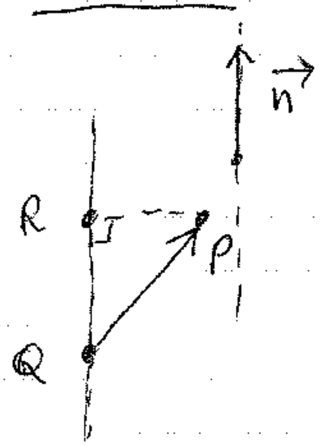
Given:

$S \Leftrightarrow$
- P in S $\hookrightarrow (x_1, y_1, z_1)$
- $\vec{n} \perp S$

Find:

$|QR|$, where $QR \perp S$

Solution:



$$\vec{QR} = \mathrm{proj}_{\vec{n}} \vec{QP}$$

$$|\vec{QR}| = \mathrm{comp}_{\vec{n}} \vec{QP}$$

$$\vec{QP} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

Substitute this into formula for comp (topic 4a in Lec. 2) + use eq. for S.

Details of calculation are in Ex. 8 in book.

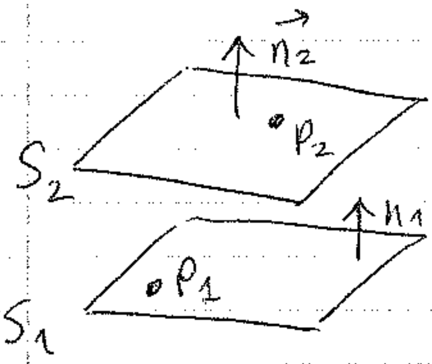
Result:

$$\boxed{D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

distance between plane $ax + by + cz + d = 0$ and pt. (x_0, y_0, z_0).

$\boxed{b}$ Distance between two $\parallel$ planes

Main point: reduce this new problem to the previous one (just as in ~~f~~topics (3b) & (3a) above).



Given: S_1 & S_2
$\Updownarrow$
(list ingredients! $P_1, \vec{n}_1$; $P_2, \vec{n}_2$)

Find: $dist(S_1, S_2)$.

Sol'n: this distance is, obviously, that between P_1 & S_2 (or vice versa); so the problem reduces to that in topic (6a). ✓

See Ex. 9 in book.

HW: Sec. 12.5.

24, 27 – standard form of eq.
33, 63 – plane thru 3 pts
45 – line $\cap$ plane
53, 55 – planes $\parallel$ or at an angle
71, 72 – dist pt. to plane; 73 – dist between $\parallel$ planes;
1 – general location of planes/lines.

EC #3: #66; Word problem about topic (6b).