

Lecture 9Euler's method

Goal: Solve $\frac{dy}{dt} = f(t, y)$ approximately
when it cannot be solved exactly.

① Idea: Approximate dy/dt :

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h} \quad \text{for small } h.$$

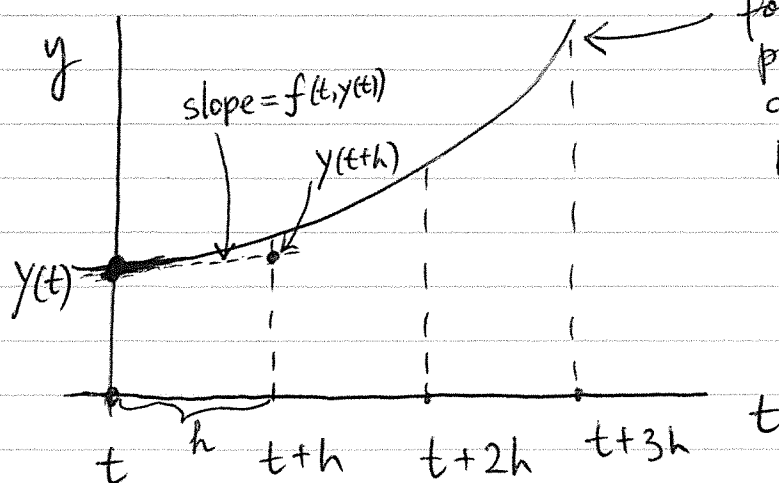
So, replace the DE with:

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t))$$

and solve for the "new" value $y(t+h)$
using the known "old" value $y(t)$:

$$\boxed{y(t+h) = y(t) + h \cdot f(t, y(t))} \quad \text{Euler's Method.}$$

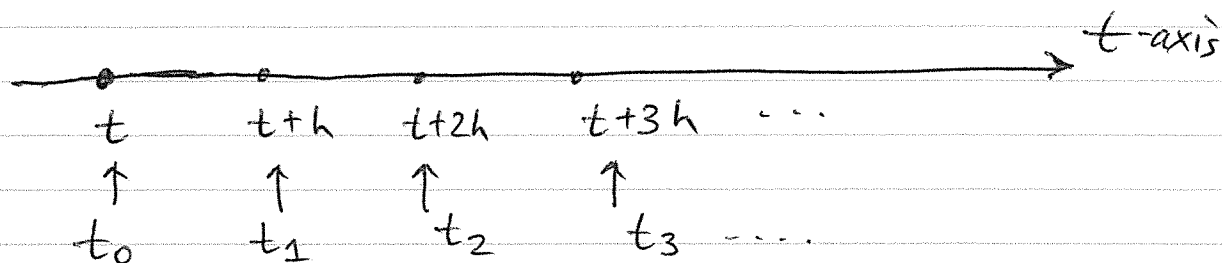
② Graphic representation



for this illustration
purpose, let's
assume that we
know the sol'n $y(t)$

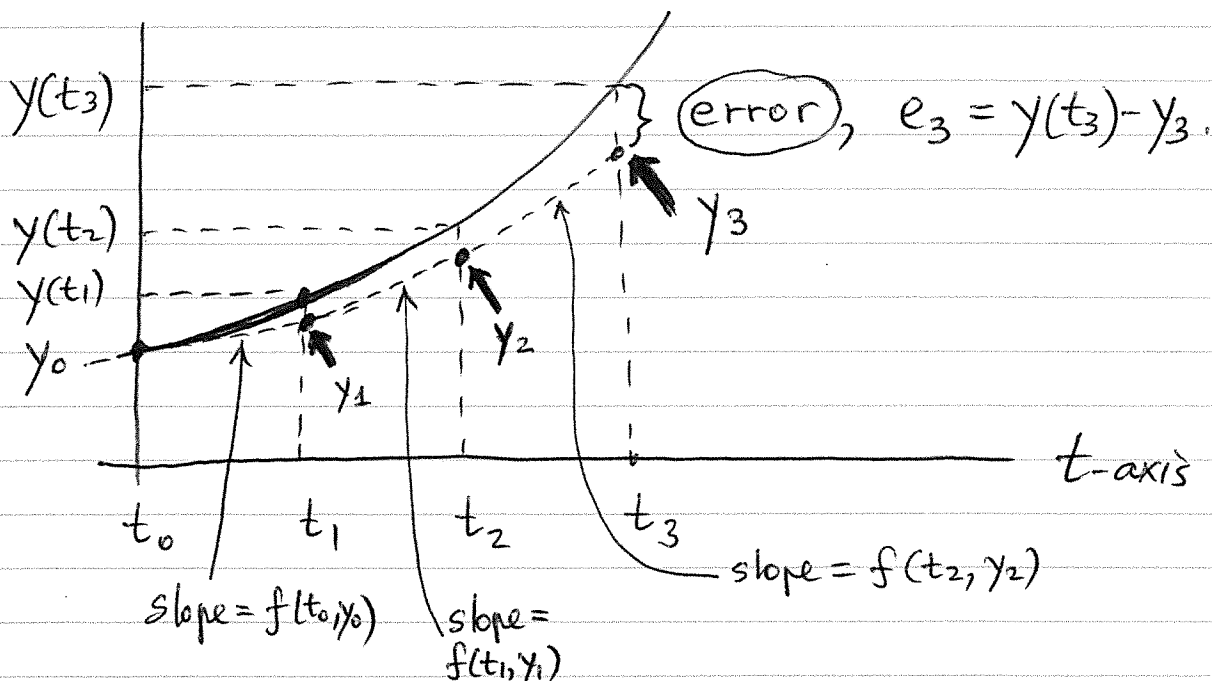
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Let's introduce a more convenient notation:



$$t_k = t_0 + k \cdot h, \quad k=0, 1, 2, \dots$$

Then we can redraw the previous figure:



Another notation: write Euler's method as

$$y_{k+1} = y_k + h \cdot f(t_k, y_k)$$

y_k = approximate solution at $t=t_k$.
 $y(t_k)$ = exact (and usually unknown) solution at $t=t_k$

Ex. 1 Compute y_1, y_2, y_3 for

$$y' = 0.5y + t, \quad y(t_0) = 3; \quad t_0 = 0;$$

while using $h = 0.1$.

Sol'n: $y_{k+1} = y_k + h \cdot f(t_k, y_k)$
 $= 3 \Rightarrow k = 2.$

Thus need to do the calculations for $k = 0, 1, 2$.

$k=0$ $y_{\underbrace{0+1}_1} = y_0 + h \cdot f(t_0, y_0)$
 $= 3 + 0.1 \cdot (0.5 \cdot \overset{3}{y_0} + \overset{0}{t_0})$ $\downarrow y_1$
 $= 3 + 0.1 \cdot (0.5 \cdot 3 + 0) = \boxed{3.15}$

$k=1$ $y_{\underbrace{1+1}_2} = y_1 + h \cdot f(t_1, y_1)$
 $= 3.15 + 0.1 \cdot (0.5 \cdot \overset{3.15}{y_1} + \overset{0.1}{t_1})$ $\downarrow y_2$
 $= 3.15 + 0.1 \cdot 1.675 = \boxed{3.3175}$

$k=2$ $y_{\underbrace{2+1}_3} = y_2 + h \cdot f(t_2, y_2)$
 $= 3.3175 + 0.1 \cdot (0.5 \cdot \overset{3.3175}{y_2} + \overset{0.2}{t_2})$ $\downarrow y_3$
 $= 3.3175 + 0.1 \cdot 1.85875 = \boxed{3.503375}$

Exact sol'n (Lec. 2): $y(\underbrace{t_3}_{0.3}) = 3.53284.$

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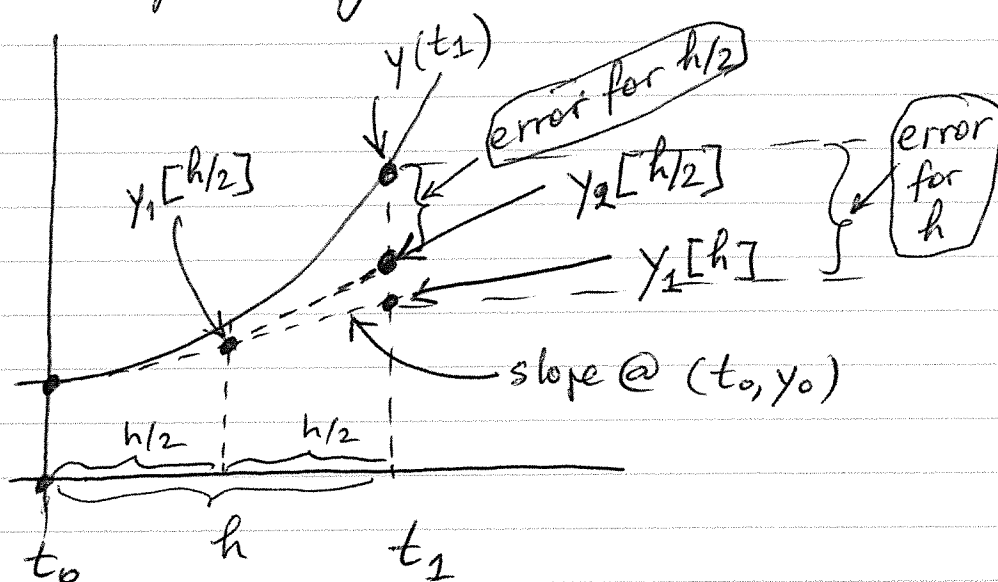
The error at $t_3 = 0.3$ is

$$|y(t_3) - y_3| = |3.53284 - 3.503375| \approx 0.0295,$$

or $\approx 0.9\%$.

Note: The error will decrease if we take a smaller h (expectedly)!

Why? Graphically:



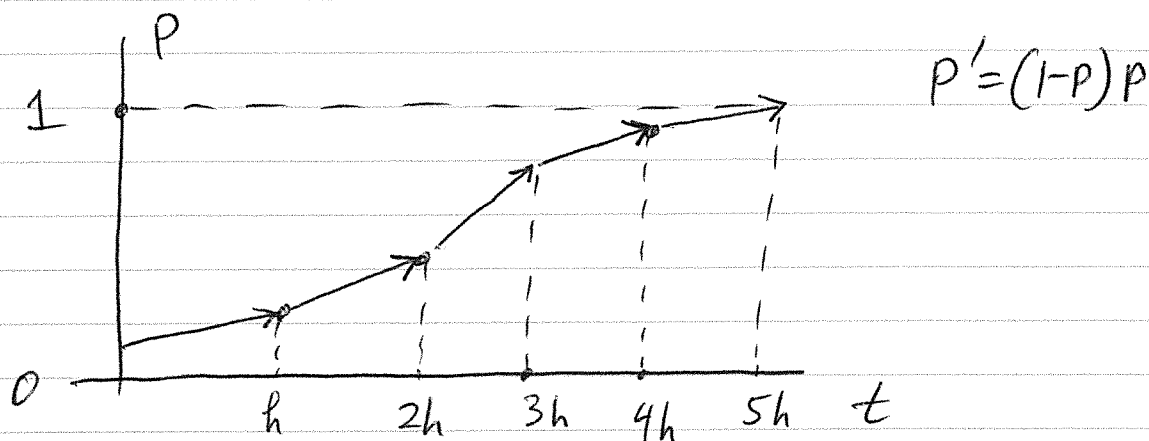
We see that $y_2[h/2]$ is a better approximation to $y(t_1)$ than $y_1[h]$.

Mnemonically: The more accurate solution $y_{2k}[h/2]$ is

always found between the exact sol'n $y(t_k)$ and the less accurate sol'n $y_k[h]$.

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③ Relation to direction field:



The approximate solution found by Euler's method follows arrows of the direction field.

HW: Sec. 2.10 ## 3, 5.

Note: 1) Do not compute y_3 & e_3 .
for both

2) In addition to the assigned calculation, repeat it for $h=0.05$ and compute y_1, y_2, y_3, y_4 for this new h , as well as e_1, e_2, e_3, e_4 .

Verify that $e_4[h=0.05] \approx \frac{1}{2} e_2[h=0.1]$.

(In general, $e_k = y(t_k) - y_k$ for a given h .)