in the same order the points were listed and formed columns using the same order the unknowns were listed in Eq. (3). For example, the third row of the augmented matrix arises from inserting (2, 2) into Eq. (3):

$$4a + 4b + 4c + 2d + 2e + f = 0.$$

In particular, the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 4 & 4 & 4 & 2 & 2 & 1 & 0 \\ 4 & -2 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 9 & 0 & -3 & 1 & 0 \end{bmatrix}.$$

We used MATLAB to transform the augmented matrix to reduced echelon form, finding

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 7/18 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -11/18 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2/3 & 0 \end{bmatrix}.$$

Thus, the coefficients of the conic through these five points are given by

$$a = -7f/18, b = f/2, c = -f/3, d = 11f/18, e = -2f/3.$$

Setting $f = 18$, we obtain a version of Eq. (3) with integer coefficients:

$$-7x^2 + 9xy - 6y^2 + 11x - 12y + 18 = 0.$$

The graph of this equation is an ellipse and is shown in Fig. 1.7. The graph was drawn using the contour command from MATLAB. Contour plots and other features of MATLAB graphics are described in the Appendix.

Finally, it should be noted that the ideas discussed above are not limited to the xy-plane. For example, consider the quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0. \qquad (4)$$

The graph of Eq. (4) is a surface in three-space; the surface is known as a ***quadric surface***. Counting the coefficients in Eq. (4), we find ten. Thus, given any nine points in three-space, we can find a quadric surface passing through the nine points (see Exercises 30–31).

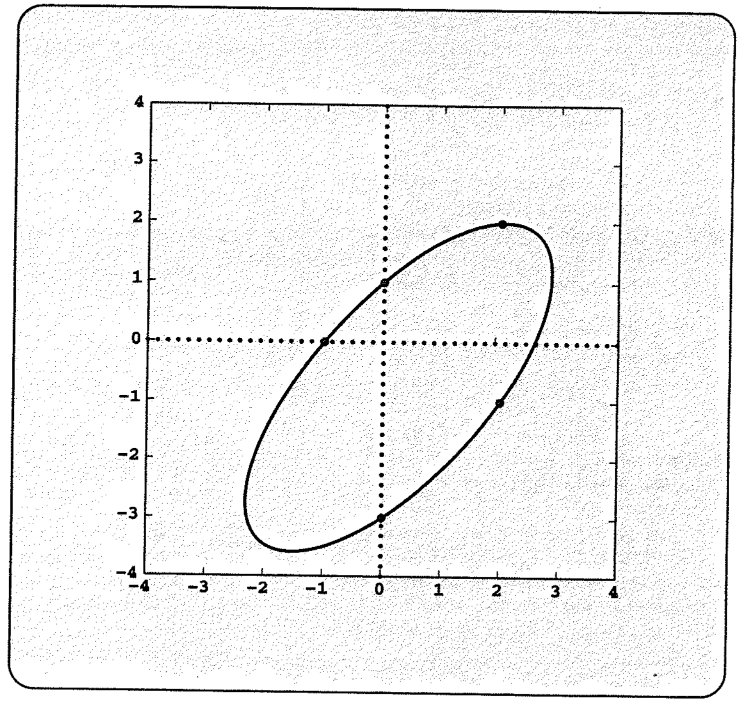

Figure 1.7 The ellipse determined by five data points, see Example 9.

1.3 EXERCISES

In Exercises 1–4, transform the augmented matrix for the given system to reduced echelon form and, in the notation of Theorem 3, determine n, r, and the number, $n - r$, of independent variables. If $n - r > 0$, then identify $n - r$ independent variables.

1. $$\begin{aligned} 2x_1 + 2x_2 - x_3 &= 1 \\ -2x_1 - 2x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 + 5x_3 &= 5 \\ -2x_1 - 2x_2 - 2x_3 &= -3 \end{aligned}$$

2. $$\begin{aligned} 2x_1 + 2x_2 &= 1 \\ 4x_1 + 5x_2 &= 4 \\ 4x_1 + 2x_2 &= -2 \end{aligned}$$

3. $$\begin{aligned} -x_2 + x_3 + x_4 &= 2 \\ x_1 + 2x_2 + 2x_3 - x_4 &= 3 \\ x_1 + 3x_2 + x_3 \quad &= 2 \end{aligned}$$

4. $$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 5x_4 &= 2 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 1 \\ -x_1 - 2x_2 - 3x_3 + 7x_4 &= 2 \end{aligned}$$

In Exercises 5 and 6, assume that the given system is consistent. For each system determine, in the notation of Theorem 3, all possibilities for the number, r of nonzero rows and the number, $n - r$, of unconstrained variables. Can the system have a unique solution?

5. $ax_1 + bx_2 = c$
$dx_1 + ex_2 = f$
$gx_1 + hx_2 = i$

6. $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$

ı Exercises 7–18, determine all possibilities for the ›lution set (from among infinitely many solutions, a ıique solution, or no solution) of the system of linear ןuations described.

7. A homogeneous system of 3 equations in 4 unknowns.
8. A homogeneous system of 4 equations in 5 unknowns.
9. A system of 3 equations in 2 unknowns.
0. A system of 4 equations in 3 unknowns.
1. A homogeneous system of 3 equations in 2 unknowns.
2. A homogeneous system of 4 equations in 3 unknowns.
3. A system of 2 equations in 3 unknowns that has $x_1 = 1, x_2 = 2, x_3 = -1$ as a solution.
4. A system of 3 equations in 4 unknowns that has $x_1 = -1, x_2 = 0, x_3 = 2, x_4 = -3$ as a solution.
5. A homogeneous system of 2 equations in 2 unknowns.
6. A homogeneous system of 3 equations in 3 unknowns.
7. A homogeneous system of 2 equations in 2 unknowns that has solution $x_1 = 1, x_2 = -1$.
8. A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1, x_2 = 3, x_3 = -1$.

ı Exercises 19–22, determine by inspection whether the iven system has nontrivial solutions or only the trivial olution.

9. $2x_1 + 3x_2 - x_3 = 0$
$x_1 - x_2 + 2x_3 = 0$

0. $x_1 + 2x_2 - x_3 + 2x_4 = 0$
$2x_1 + x_2 + x_3 - x_4 = 0$
$3x_1 - x_2 - 2x_3 + 3x_4 = 0$

1. $x_1 + 2x_2 - x_3 = 0$
$x_2 + 2x_3 = 0$
$4x_3 = 0$

2. $x_1 - x_2 = 0$
$3x_1 = 0$
$2x_1 + x_2 = 0$

23. For what value(s) of a does the system have nontrivial solutions?

$$x_1 + 2x_2 + x_3 = 0$$
$$-x_1 - x_2 + x_3 = 0$$
$$3x_1 + 4x_2 + ax_3 = 0.$$

24. Consider the system of equations

$$x_1 + 3x_2 - x_3 = b_1$$
$$x_1 + 2x_2 = b_2$$
$$3x_1 + 7x_2 - x_3 = b_3.$$

a) Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system to be consistent. [*Hint:* Reduce the augmented matrix for the system.]

b) In each of the following, either use your answer from **a)** to show the system is inconsistent or exhibit a solution.

i) $b_1 = 1, b_2 = 1, b_3 = 3$
ii) $b_1 = 1, b_2 = 0, b_3 = -1$
iii) $b_1 = 0, b_2 = 1, b_3 = 2$

25. Let B be a (4×3) matrix in reduced echelon form.

a) If B has three nonzero rows, then determine the form of B. (Using Fig. 1.5 of Section 1.2 as a guide, mark entries that may or may not be zero by $*$.)

b) Suppose that a system of 4 linear equations in 2 unknowns has augmented matrix A, where A is a (4×3) matrix row equivalent to B. Demonstrate that the system of equations is inconsistent.

In Exercises 26–31, follow the ideas illustrated in Examples 8 and 9 to find the equation of the curve or surface through the given points. For Exercises 28–29, display the graph of the equation as in Fig. 1.7.

26. The line through (3, 1) and (7, 2).
27. The line through (2, 8) and (4, 1).
28. The conic through $(-4, 0)$, $(-2, -2)$, $(0, 3)$, $(1, 1)$, and $(4, 0)$.
29. The conic through $(-4, 1)$, $(-1, 2)$, $(3, 2)$, $(5, 1)$, and $(7, -1)$.
30. The quadric surface through (0, 0, 1), (1, 0, 1), (0, 1, 0), (3, 1, 0), (2, 0, 4), (1, 1, 2), (1, 2, 1), (2, 2, 3), (2, 2, 1).
31. The quadric surface through (1, 2, 3), (2, 1, 0), (6, 0, 6), (3, 1, 3), (4, 0, 2), (5, 5, 1), (1, 1, 2), (3, 1, 4), (0, 0, 2).

In Exercises 32–33, note that the equation of a circle has the form

$$ax^2 + ay^2 + bx + cy + d = 0.$$

Hence a circle is determined by three points. Find the equation of the circle through the given points.

32. (1, 1), (2, 1), and (3, 2)
33. (4, 3), (1, 2), and (2, 0)

1.4 APPLICATIONS (OPTIONAL)

In this brief section we discuss networks and methods for determining flows in networks. An example of a network is the system of one-way streets shown in Fig. 1.8. A typical problem associated with networks is estimating the flow of traffic through this network of streets. Another example is the electrical network shown in Fig. 1.9. A typical problem consists of determining the currents flowing through the loops of the circuit.

(*Note:* The network problems we discuss in this section are kept very simple so that the computational details do not obscure the ideas.)

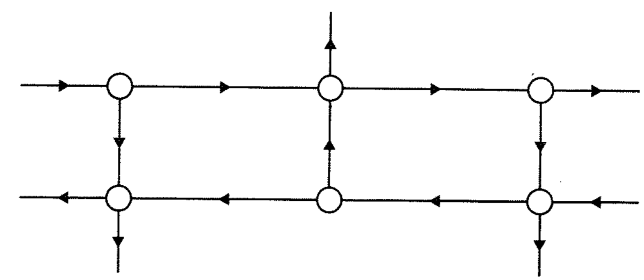

Figure 1.8 A network of one-way streets

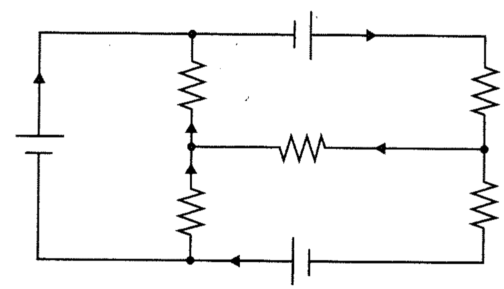

Figure 1.9 An electrical network