

Sec. 3.6 : Orthogonal bases

14-1

① Why are orthogonal bases better than generic bases?

In \mathbb{R}^2 , vectors are orthogonal (perpendicular) when their dot product is 0:

$$(\underline{v}_1 \perp \underline{v}_2) \Leftrightarrow (\underline{v}_1^T \underline{v}_2 = 0).$$

The same definition carries over to \mathbb{R}^n :

Def: A set $\{\underline{v}_1, \dots, \underline{v}_p\}$ of vectors in \mathbb{R}^n is orthogonal if each pair of distinct vectors

is orthogonal: $\underline{v}_i^T \underline{v}_j = 0$ for $i \neq j$.

See Ex. 1 in book for numbers.

Note: Our intuitive notion that

$(\underline{v}_1 \perp \underline{v}_2) \Leftrightarrow (\underline{v}_2 \perp \underline{v}_1)$ is supported by the above definition. I.e., we want to show:

$$(\underline{v}_1^T \underline{v}_2 = 0) \Rightarrow (\underline{v}_2^T \underline{v}_1 = 0). \text{ Indeed:}$$

$$(\underline{v}_1^T \underline{v}_2 = 0)^T \Rightarrow (\underline{v}_1^T \underline{v}_2)^T = 0^T \Rightarrow \underline{v}_2^T (\underline{v}_1^T)^T = 0$$

$$\Rightarrow \underline{v}_2^T \underline{v}_1 = 0. \quad \checkmark$$

↑
Thm. 10 of
Chap. 1

↑
scalar

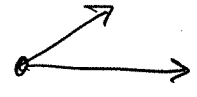
Thm. 13: If $\{ \underline{v}_1, \dots, \underline{v}_p \}$ = orthogonal set
 $\{ \underline{v}_1, \dots, \underline{v}_p \} \Rightarrow$ lin. indep. set

Illustration:



← This is clearly lin. indep.

Note that ("lin. indep.") $\not\Rightarrow$ "orthogonal" in general:



← This is lin. indep., but not orthogonal.

Given:

$$\underline{v}_i^T \underline{v}_j = 0 \text{ for } i \neq j$$

Want:

$$c_1 \underline{v}_1 + \dots + c_p \underline{v}_p = \underline{0} \\ \Rightarrow c_1 = \dots = c_p = 0.$$

Proof:

$$1) \underline{v}_1^T (c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_p \underline{v}_p = \underline{0}) \\ c_1 (\underline{v}_1^T \underline{v}_1) + c_2 (\underline{v}_1^T \underline{v}_2) + \dots + c_p (\underline{v}_1^T \underline{v}_p) = (\underline{v}_1^T \underline{0}) \rightarrow 0 \\ \|\underline{v}_1\|^2 \leftarrow \text{Sec. 1.6, p. 68} \quad 0 \leftarrow \text{Given} \rightarrow 0$$

$$c_1 \underbrace{\|\underline{v}_1\|^2}_{\neq 0} + 0 + \dots + 0 = 0 \Rightarrow c_1 = 0$$

2) If we multiply by \underline{v}_2^T , we get $c_2 = 0$.

Similarly, all $c_1 = c_2 = \dots = c_p = 0$.

q.e.d.

Claim:

It is much easier to find coordinates of a vector in an orthogonal basis than in a generic basis.

- To find coordinates of \underline{x} in a generic basis $\{\underline{v}_1, \dots, \underline{v}_p\}$:

$$\underline{x} = c_1 \underline{v}_1 + \dots + c_p \underline{v}_p$$

$$\underline{x} = [\underline{v}_1, \dots, \underline{v}_p] \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$$

$$\underline{V} \underline{c} = \underline{x} \Rightarrow \text{solve by REF.}$$

Amount of calculation grows rapidly with p .

- To find coordinates of \underline{x} in an orthogonal basis:

$$\underline{v}_1^T (c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_p \underline{v}_p = \underline{x})$$

$$c_1 (\underline{v}_1^T \underline{v}_1) + c_2 (\underline{v}_1^T \underline{v}_2) + \dots + c_p (\underline{v}_1^T \underline{v}_p) = \underline{v}_1^T \underline{x}$$

$$\underbrace{\|\underline{v}_1\|^2}_0 + \underbrace{0}_0 + \dots + \underbrace{0}_0 \leftarrow \text{As in the Proof of Thm. 13}$$

$$\Rightarrow \boxed{c_1 = \frac{\underline{v}_1^T \underline{x}}{\|\underline{v}_1\|^2}} \quad \text{Similarly:} \quad \boxed{c_2 = \frac{\underline{v}_2^T \underline{x}}{\|\underline{v}_2\|^2}} \quad \dots \quad \boxed{c_p = \frac{\underline{v}_p^T \underline{x}}{\|\underline{v}_p\|^2}}$$

Coordinates of \underline{x} in an orthogonal basis.

See Ex. 4 in book for numbers.

A simplification occurs if the lengths of all vectors in an orthogonal basis is 1. Hence:

Def: An orthonormal basis is:
 "⊥" "length"

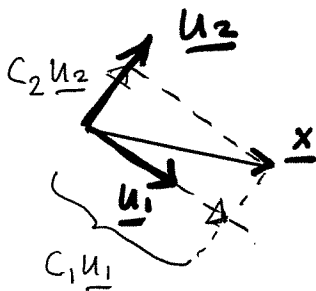
- an orthogonal basis, where
- lengths of all basis vectors is 1:
 $\|\underline{u}_i\| = 1$ for $i = 1, \dots, p$.

② Projection of \underline{x} on \underline{v}

Def: Let $\{ \underline{v}_1, \dots, \underline{v}_p \}$ be an orthogonal basis; then $\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_p \underline{v}_p$.

Projections of \underline{x} on $\underline{v}_1, \dots, \underline{v}_p$ $\rightarrow P_{\underline{v}_1}(\underline{x}) \quad P_{\underline{v}_2}(\underline{x}) \quad \dots \quad P_{\underline{v}_p}(\underline{x})$

• Aside note: Geometric meaning of coordinates in an orthonormal basis:



They are the lengths of projections of \underline{x} on the unit basis vectors $\underline{u}_1, \dots, \underline{u}_p$.

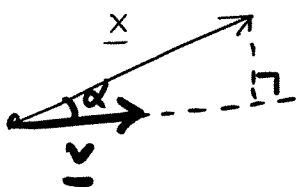
• Above we have derived formulas for coordinates in an orthogonal basis. So we can now write formulas for projections:

$$P_{\underline{v}_1}(\underline{x}) = c_1 \underline{v}_1 = \frac{\underline{v}_1^T \underline{x}}{\|\underline{v}_1\|^2} \underline{v}_1, \text{ etc.}$$

We can write the same formula for projection of \underline{x} on any one given vector \underline{v} :

$$P_{\underline{v}}(\underline{x}) = \frac{\underline{v}^T \underline{x}}{\|\underline{v}\|^2} \cdot \underline{v}$$

Derivation based on Calculus: $P_{\underline{v}}(\underline{x}) = \underbrace{\|\underline{x}\| \cos \alpha}_{\text{length}} \cdot \underbrace{\left(\frac{\underline{v}}{\|\underline{v}\|} \right)}_{\text{unit vector along } \underline{v}}$

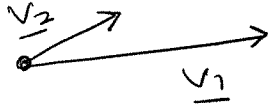


$$= \frac{\underbrace{\|\underline{x}\| \cdot \|\underline{v}\| \cdot \cos \alpha}_{\text{dot product of } \underline{x} \text{ \& } \underline{v}}}{\|\underline{v}\|} \cdot \frac{\underline{v}}{\|\underline{v}\|} = \frac{\underline{v}^T \underline{x}}{\|\underline{v}\|} \cdot \frac{\underline{v}}{\|\underline{v}\|}$$

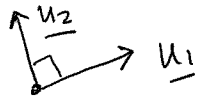
③ How to construct an orthonormal basis from a generic basis
 (The Gram-Schmidt orthogonalization)

In \mathbb{R}^2 :

Given:



Want:

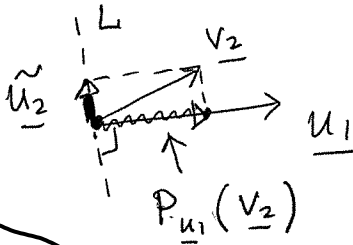


- $\underline{u}_1 \perp \underline{u}_2$
- $\|\underline{u}_1\| = \|\underline{u}_2\| = 1$

• $\underline{u}_1, \underline{u}_2$ are "made" from $\underline{v}_1, \underline{v}_2$.

Step 1: Construct \underline{u}_1 : $\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$

Step 2: Construct \underline{u}_2 :



• Draw line $L \perp \underline{u}_1$

• Write \underline{v}_2 as: ALONG line L

$$\underline{v}_2 = \underbrace{P_{\underline{u}_1}(\underline{v}_2)}_{\substack{\text{given} \\ \text{find using} \\ \text{formula}}} + \underbrace{\tilde{\underline{u}}_2}_{\text{want}}$$

Simplified formula

$\Rightarrow \tilde{\underline{u}}_2 = \underline{v}_2 - P_{\underline{u}_1}(\underline{v}_2)$; $\tilde{\underline{u}}_2 \perp \underline{u}_1$ by design.

So, when $\|\underline{u}\| = 1$,

$P_{\underline{u}}(\underline{v}) = \underline{u} \cdot (\underline{u}^T \underline{v})$.

$$\frac{\underline{u}_1^T \underline{v}_2}{\|\underline{u}_1\|^2} \cdot \underline{u}_1 \equiv \underbrace{(\underline{u}_1^T \underline{v}_2)}_{\text{since } \|\underline{u}_1\|=1} \cdot \underline{u}_1$$

• Make a unit \underline{u}_2 : $\underline{u}_2 = \tilde{\underline{u}}_2 / \|\tilde{\underline{u}}_2\|$.

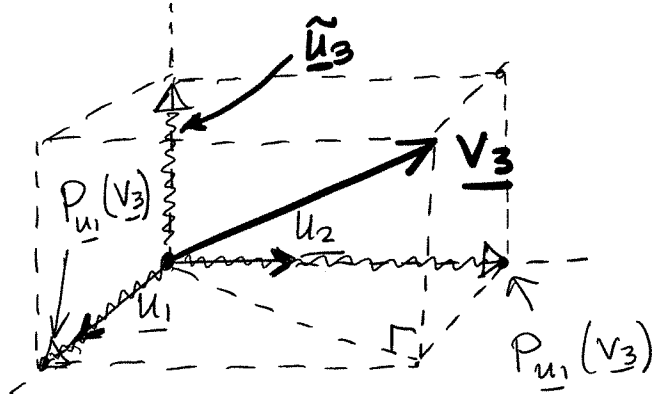
In \mathbb{R}^3 : Given basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, construct an orthonormal basis $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$.

Step 1: Same as in \mathbb{R}^2 : $\underline{u}_1 = \underline{v}_1 / \|\underline{v}_1\|$.

Step 2: \underline{v}_2 and \underline{u}_1 are in the same plane (because any two vectors are always in the same plane). Therefore, can apply the same process as in \mathbb{R}^2 to "make" \underline{u}_2 :

- $\underline{\tilde{u}}_2 = \underline{v}_2 - P_{\underline{u}_1}(\underline{v}_2)$
- $\underline{u}_2 = \underline{\tilde{u}}_2 / \|\underline{\tilde{u}}_2\|$.

Step 3 We now have 2 unit orthogonal vectors \underline{u}_1 & \underline{u}_2 , and we can think of them as "our" \vec{i} and \vec{j} (unit coordinate vectors in 3D).



Write
 $\underline{v}_3 = P_{\underline{u}_1}(\underline{v}_3) + P_{\underline{u}_2}(\underline{v}_3) + \underline{\tilde{u}}_3$
 ↑ Given ↑ formula ↑
 ⊥ to plane made by $\underline{u}_1, \underline{u}_2$

$\Rightarrow \underline{\tilde{u}}_3 = \underline{v}_3 - P_{\underline{u}_1}(\underline{v}_3) - P_{\underline{u}_2}(\underline{v}_3)$

- Make a unit \underline{u}_3 : $\underline{u}_3 = \underline{\tilde{u}}_3 / \|\underline{\tilde{u}}_3\|$.

Note: Ex. 5 & 6 in book use an equivalent process, but they do not normalize their vectors $\underline{\tilde{u}}_2, \underline{\tilde{u}}_3$ etc. (and do not use the "tilde"). You may use either the Notes' or the book's process.