

**Reminder:** *You MUST show your work to get credit.*

## Project 5: Visualizing eigenvalues and eigenvectors in $\mathbb{R}^2$

### Goal

Develop an understanding of the geometrical meaning of eigenvectors and eigenvalues of  $2 \times 2$  matrices.

### General requirements

- You may work alone or with *one* other person. If you work with someone else, hand in *one* answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups. **Please read “My policies on Projects” posted on the course website.**
- Write your answers on the answer sheet provided provided in the last few pages of this document. Staple (with a staple, not paper clip)<sup>1</sup> all the paper showing your *neatly presented*<sup>2</sup> work to the answer sheet.

### Introduction

Eigenvalues and eigenvectors arise in a great variety of problems in science and engineering. In particular, when matrices are used to describe a given system, the eigenvalues and eigenvectors determine the *long-term behavior* of that system. One example was worked out in the Lecture for Sec. 4.1, where a certain eigenvector predicted an equilibrium distribution of subscribers and non-subscribers to a local newspaper in a town. Another example where you should have observed a similar behavior was in Exercise 2 of Project 1, where the numbers of donating and non-donating alumni reached constant values. Curiously, the key idea behind an early version<sup>3</sup> of Google’s data sorting algorithm is also based on analyzing a distribution of components of a certain eigenvector. There are literally thousands of diverse manifestations of eigenvalues and eigenvectors in Nature and even in our everyday lives.

Given the paramount importance of the eigenvalues and eigenvectors, most software packages have built-in capabilities to compute them. It should be noted that the algorithms used to find the eigenvectors and eigenvalues do **not** use determinants and characteristic polynomials; see # 31 for Sec. 4.2 as to why. The practical algorithms for finding eigenvalues and eigenvectors are studied in courses on Numerical Analysis.

In this project, you will study eigenvalues and eigenvectors of  $2 \times 2$  matrices that represent linear transformations in  $\mathbb{R}^2$ . There are **two main intended outcomes** of this study.

- You will discover that a composition of stretching/shrinking and rotations can sometimes amount to merely stretching or shrinking along certain directions, determined by eigenvectors. A prominent example of this situation occurs in Mechanics of materials, where a combination of pure stresses and shear can be viewed as stresses (or a stress and a strain) along so called principal axes.
- You will then (re)discover that repeated application of the same transformation “favors” directions along eigenvectors corresponding to certain eigenvalues, while “downplaying” directions along other eigenvectors. This is the situation that has been mentioned above in connection with Project 1 and Sec. 4.1.

This project makes heavy use of Matlab. You should use the Matlab resources posted on the course webpage. You may also use other resources or Matlab’s online help.

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<sup>1</sup>Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.

<sup>2</sup>I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.

<sup>3</sup>and possibly more recent versions, too

## Exercise 1

In this Exercise, you will find a matrix that describes the consecutive action of two linear transformations.

(a) Copy the code which drew your initial letter in Project 4 into a new file. Create a matrix with the letter's  $x$ - and  $y$ -coordinates and name it `original_letter`.

(b) Write a code which applies to this letter two transformations (see below), in the specified order.

- The first transformation stretches the letter horizontally by the factor of 2 and shrinks it vertically by the same factor.
- The second transformation rotates all points in  $\mathbb{R}^2$  by  $\pi/6$  radians about the origin.
- Name this twice-transformed letter `transformed_letter_Ex1b`. Do not plot it *yet*; you will be asked to do so in part (c) below.

*Note 1:* You have not previously derived a matrix of a stretching or shrinking transformation, but this is easy to do. First, from the description, write down what the transformation does to each component of  $\underline{x} = [x_1, x_2]^T$ . (You may find inspiration in Example 8 in posted Notes for Sec. 3.7.) Then proceed as you did in Exercise 1 of Project 1. You should obtain a simple, diagonal  $2 \times 2$  matrix.

*Note 2:* The matrix of rotation was derived by you in Exercise 1 of Project 1; see also p. 236 of the textbook.

*Note 3:* Your Matlab code applying a transformation to the letter matrix (i.e., to the `original_letter` defined above) should mimic that in Part 2 of Project 4. *See its posted solution* if you could not do that part yourself.

(c) Write down a single matrix which represents the *composition* of the above two transformations in the order specified in part (b). Present a brief calculation, or give a Matlab command, showing how you have found this matrix. In what follows, I will refer to this transformation as  $T_C$  and to its matrix as  $C$  (for Composition).

Transform your original letter (i.e., the matrix `original_letter`) with matrix  $C$  and call the resulting matrix `transformedbyC_letter`. Make a plot of the `transformedbyC_letter` and include it in your work.

To check that your composition matrix is correct, plot the letter from part (b), i.e., `transformed_letter_Ex1b`, in the same plot with the `transformedbyC_letter`. Use different line styles<sup>4</sup> (not colors!) for the two letters.

Finally, state whether your composition transformation  $T_C$  is linear; briefly give a reason.

*Note 4:* The material on composition of two transformations is found in the posted Notes for Sec. 3.7.

*Note 5:* Since `transformed_letter_Ex1b` and `transformedbyC_letter` are supposed to be identical, it will be impossible to tell them apart in the same plot. To distinguish them and yet be able to see that they are the same, add a small scalar, say 0.005, to each entry of `transformedbyC_letter`. Here Matlab's convention that  $A + r$  means adding the same scalar  $r$  to each entry of matrix  $A$  will come very handy.

## Exercise 2

Transformation  $T_C$  is a *composition* of stretching/shrinking along two orthogonal directions (horizontal and vertical) and a rotation. In this Exercise, you will demonstrate that it is equivalent to *just stretching and shrinking*, but along two non-orthogonal directions. These directions will turn out to be those of the eigenvectors of the matrix  $C$ .

### Preamble

To motivate your work in this Exercise, let us recall that in Sec. 4.7 we showed that diagonalization of a matrix,  $C = VDV^{-1}$  in this case, can be written in an equivalent form:

$$CV = VD, \quad V \equiv [\underline{v}_1, \underline{v}_2], \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \quad (1a)$$

where  $\underline{v}_1, \underline{v}_2$  are linearly independent eigenvectors of  $C$ :

$$C\underline{v}_1 = \lambda_1\underline{v}_1, \quad C\underline{v}_2 = \lambda_2\underline{v}_2. \quad (1b)$$

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<sup>4</sup>Search for 'Line specification in Matlab' online.

Let us note that (1b) is consistent with (1a) also in the following sense:

$$(1b) \iff [C\underline{\mathbf{v}}_1, C\underline{\mathbf{v}}_2] = [\lambda_1\underline{\mathbf{v}}_1, \lambda_2\underline{\mathbf{v}}_2] \iff C[\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] = [\lambda_1\underline{\mathbf{v}}_1 + 0\underline{\mathbf{v}}_2, 0\underline{\mathbf{v}}_1 + \lambda_2\underline{\mathbf{v}}_2]$$

$$\xLeftrightarrow{\text{Key formula}} C[\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] = [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \iff (1a). \quad (1c)$$

Let us now recall how (1a) can be used to interpret the transformation of an arbitrary vector  $\underline{\mathbf{u}}$  in  $\mathbb{R}^2$  by matrix  $C$ . Since  $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$  are linearly independent, they form a basis in  $\mathbb{R}^2$ , and one can expand  $\underline{\mathbf{u}}$  over that basis:

$$\underline{\mathbf{u}} = \underline{\mathbf{v}}_1 c_1 + \underline{\mathbf{v}}_2 c_2 = V \underline{\mathbf{c}}; \quad \underline{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \quad (2)$$

here  $c_1, c_2$  are the coordinates of  $\underline{\mathbf{u}}$  in the basis  $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2\}$ . The transformation of  $\underline{\mathbf{u}}$  by matrix  $C$  can be viewed in that basis as follows:

$$C\underline{\mathbf{u}} \stackrel{\text{Step 1}}{=} C V \underline{\mathbf{c}} \stackrel{\text{Step 2}}{=} V D \underline{\mathbf{c}} \stackrel{\text{Step 3}}{=} [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} \stackrel{\text{Step 4}}{=} \underline{\mathbf{v}}_1 \lambda_1 c_1 + \underline{\mathbf{v}}_2 \lambda_2 c_2. \quad (3)$$

In Step 1, we have written  $\underline{\mathbf{u}}$  in terms of its coordinates in the eigenbasis (i.e., “own” basis) of  $C$ ; see (2).

In Step 2, we have used how  $C$  transforms its “own” basis; see (1a) and (1c).

In Step 3, the “eigencore” of  $C$ , i.e., the diagonal matrix  $D$ , transforms the coordinates.

Finally, in Step 4, these transformed coordinates are used to represent  $C \underline{\mathbf{u}}$  in the eigenbasis of  $C$ .

### Assignment

(a) Use the command `[V,D] = eig(C)` to find the eigenbasis and the eigenvalues of  $C$ ; type `help eig` for details.

(b) Implement Step 1 of the Preamble. That is, find the coordinates of each point of your letter in the eigenbasis of  $C$  by solving the counterpart of Eq. (2).

*Note 1:* Here,  $\underline{\mathbf{u}}$  will be not a vector but a matrix (i.e., a collection of column vectors) representing your letter, defined in Exercise 1(a). See the Example before Part 2 of Project 4 about representing the matrix of a letter as a collection of vectors. Thus, instead of one column vector  $\underline{\mathbf{c}}$ , as in Eq. (2), you will have a matrix, representing *the collection* of coordinates of your letter in the eigenbasis of  $C$ .

*Note 2:* You can use any of Matlab’s commands for solving a linear system, such as the backslash (`help \`) or `inv`. Note that the more familiar command `rref` is *not* a convenient option in this case.

(c) Implement Step 3 of the Preamble. That is, multiply the coordinates by their respective eigenvalues.

*Note 3:* You do *not* need to extract the eigenvalues from matrix  $D$ , obtained in part (a) of this Exercise; nor do you need to break your matrix of coordinates into its rows. *Hint:* Count the dimensions of these matrices and look at Step 3 in Eq. (3).

(d) Implement Step 4 of the Preamble. That is, write the transformed matrix of your letter as the linear superposition of the eigenvectors of  $C$ . Name the result `transformed2_letter`.

*Note 4:* Again, you do *not* need to break either  $V$  into its columns or the matrix of transformed coordinates into its rows. See the *Hint* above.

(e) Finally, compare your `transformed2_letter` with `transformedbyC_letter`, which was the result of *direct* transformation of `original_letter` by matrix  $C$  (see Exercise 1(c)). For that, plot `transformed2_letter` and `transformedbyC_letter` in the same plot using different line styles (not colors!) for the two letters.

*Note 5:* Since `transformed2_letter` and `transformedbyC_letter` are supposed to be identical, it will be impossible to tell them apart in the same plot. Use the trick described in Note 5 for Exercise 1 to address this issue.

### Exercise 3

Here you will explore a relation between a *repeated* transformation and eigenvalues and eigenvectors of the corresponding matrix.

(a) Find the result of transforming your original letter 5 times with the composition transformation  $T_C$ , repeating the steps of obtaining `transformedbyC_letter` in part (c) of Exercise 1.

Plot the result. Set the ranges along the horizontal and vertical axes of your plot equal by inserting the command `axis('equal')` after the plot, but do *not* impose specific limits on the axes.

“Freeze” the figure window by typing `hold on`; this will be needed to combine this plot with another one in part (b) below. Type `help hold` if you need this command explained.

*Note 1:* In carrying out a scientific investigation, one should try to follow two *general principles*:

(i) Always proceed gradually, not in leaps; and (ii) Reflect on the results observed. In the context of this Exercise, these principles translate into the following:

- Instead of looking at the end result right away, inspect how the letter to which  $T_C$  was applied twice differs from the letter to which  $T_C$  was applied only once. Then compare this to the letter to which  $T_C$  was applied three times, and reflect on the pattern observed. If you are still unsure, re-read the second bullet point of the Introduction.

(b) In the “frozen” figure window of part (a), plot the eigenvectors of  $C$ , which you found in Exercise 2(a).

How is the *appearance* of your final letter in part (a) of this Exercise related to the eigenvalue(s) and eigenvector(s) of  $C$ ?

*Note 2a:* A method to plot a vector (as segment of a line) was shown in Part 3 of Project 4.<sup>5</sup>

*Note 2b:* Do not hardcode (i.e., type by hand) the components of the vectors. Instead, extract them as explained in Sec. A.4 of Appendix A of the textbook or in Sec. 11 of Sigmon’s Matlab Primer.

*Note 3:* As you remember, if  $\underline{v}$  is an eigenvector of  $A$ , so is  $r \underline{v}$  for any scalar  $r$ . Matlab chooses this scalar so that the length of all eigenvectors found by command `eig` is 1. However, to make the plotted eigenvectors commensurate in size with the axes’ ranges in the plot in part (a), multiply the eigenvectors by  $r = 5$  (or something close).

(c) Compare your answers in parts (a) and (b) with the result of Exercise 2(d) in Project 1 (not 2(d) in this Project!) and with the derivation in Example 1 in the posted Notes for Sec. 4.1. **Based on your analysis of the similarities of these situations**, fill in the blanks in the text in the Answer sheet.

### Exercise 4

Suppose that instead of transformation  $T_C$  one applies another transformation,  $T_A$ , which differs from  $T_C$  *only* by the order in which stretching/shrinking and rotation are performed. That is, in this new transformation  $T_A$ , rotation is performed first, and stretching/shrinking second.

Suppose that in Exercise 2(e), you have been asked to transform your letter with  $T_A$  instead of  $T_C$ . Without doing any calculations, predict if one should expect, *on general grounds*, if the results here and in Exercise 2(e) will look different. (You do not need to worry about the addition of the small number 0.005 mentioned in Exercise 2(e).)

*Note 1:* Here “different” means that the dimensions and/or orientation of the transformed letters differ by, say, 5% or more.

*Note 2:* You may, of course, apply  $T_A$  to your letter<sup>6</sup> and use your observations in your answer. However, full credit will be given only if you base your answer on a theoretical fact covered in this course, not on your numerical experimentation. This fact is not mentioned in Chap. 3 of the textbook but *is* mentioned explicitly in the posted Notes for Sec. 3.7.

<sup>5</sup>Matlab has other commands for plotting vectors, but those require having special toolboxes.

<sup>6</sup>Just make sure not to mess up your existing code. Save the code that you plan to submit, then save a copy of it under a new name and make all modifications there, not in the original code. You will not need to submit this new code; it is for your use only.

**Bonus** Parts (h) – (k) do *not* rely on parts (c) – (g).

Credit for each part will be given if the work for that part is more than 60% correct.

- (a) Review Lecture Notes for Secs. 3.6 and 3.7 to recognize the action of  $C^k$  with  $k \geq 5$  as being similar to a familiar transformation. Describe that transformation in reasonable detail (but without doing any calculations yet).
- (b) Using Lecture Notes for Secs. 3.6 and 3.7, compute the matrix of the transformation that you have identified in part (a). Verify whether it is indeed close to  $C^k$ ,  $k \geq 5$ . (A follow-up on this question is found in a later part of this assignment.)
- (c) Now consider repeated action of the pure rotation  $T_R$  with the same rotation matrix  $R$  as in Exercise 1. Make one plot where you show several results of repeated application of  $T_R$  to your initial (once, twice,  $\dots$ , five times,  $\dots$ , until the pattern becomes obvious). Describe in words how the repeated action of  $T_R$  differs from that of the repeated action of  $T_C$ .
- (d) What *numerical characteristic(s)* of matrices  $R$  and  $C$  makes (or make) the repeated actions of their respective transformations so different? (This question goes partially beyond the material covered in this course. However, based on what you have learned in this course, you should still be able to answer that question — or at least state your answer as a plausible hypothesis — by invoking some simple logic.)

- (e) Now, consider another composition transformation,  $T_B$ , where the first transformation is a lengthening along  $x$  by a factor  $b$  and a shortening by the same factor along  $y$ , and the second transformation is the same rotation,  $T_R$ , as before. (Denote the matrix of transformation  $T_B$  as  $B$ .) One can consider a family of such compositions; then the family member with  $b = 2$  is  $T_C$  and the family member with  $b = 1$  is just  $T_R$ .

Set the value  $b = 1.5$  and have Matlab compute the corresponding  $T_B \equiv T_{1.5}$ . Similarly to what you did in (c), make a plot showing the repeated action of this  $T_B$  on your letter. Describe the observed pattern and compare it with the repeated actions of the “ $T_B$ -family” members with  $b = 2$  and  $b = 1$ .

- (f) By experimentation, find the *exact* value of  $b$ ,  $b_{\text{thresh}}$ , which demarcates the two behaviors mentioned in part (c). (I.e., one behavior is “like  $T_C$ ” and the other is “like  $T_R$ ”.) What seems to be the result of the repeated action of  $T_B$  for  $b = b_{\text{thresh}}$ ?

*Suggestion:* To find  $b_{\text{thresh}}$ , begin by varying the first decimal place of  $b$ . Once you find where the transition between the two behaviors occurs, begin varying only the second decimal place. Then you may repeat this process for the third decimal place, but by then you may already guess what  $b_{\text{thresh}}$  is.

- (g) If you answered the question in part (d), confirm your empirically found value of  $b_{\text{thresh}}$  by an analytical calculation. Provide adequate explanation for all steps of your work.

- (h) Now let us come back to the matrix  $C^k$  from part (b). Your answer in part (a) must have led you to think that  $C^k$  should be close to the form  $\underline{\mathbf{w}} \underline{\mathbf{w}}^T$  for a certain  $\underline{\mathbf{w}}$ . However, visual inspection of, say,  $C^{10}$  should convince you that this is nowhere close to  $\underline{\mathbf{w}} \underline{\mathbf{w}}^T$  no matter what  $\underline{\mathbf{w}}$  is. Explain, using a result from topic 6 of Lecture Notes for Sec. 1.6, why the previous sentence must be correct.

- (i) So, what is going on? I.e., if, according to part (a), the action of  $C^k$  *looks like* that of  $\underline{\mathbf{w}} \underline{\mathbf{w}}^T$ , then how come  $C^k$  is not anywhere close to  $\underline{\mathbf{w}} \underline{\mathbf{w}}^T$ ? The answer is:  $C^k \approx \underline{\mathbf{z}} \underline{\mathbf{z}}^T$  for some  $\underline{\mathbf{z}} \neq \underline{\mathbf{w}}$ .

Assignments for this part:

- 1) State what  $\underline{\mathbf{w}}$  is. (This is an easy question if you did Exercise 3 and part (a) above.)
- 2) Prove, by acting with  $\underline{\mathbf{w}} \underline{\mathbf{z}}^T$  on an arbitrary vector  $\underline{\mathbf{x}}$ , that your statement about  $\underline{\mathbf{w}}$  is correct.

- (j) The information given is not sufficient to find  $\underline{\mathbf{z}}$ , though. However, you can still find  $\underline{\mathbf{z}}^T \underline{\mathbf{w}}$ . You will do so in part (k). For now, let  $\underline{\mathbf{z}}^T \underline{\mathbf{w}} = s$  for some  $s$ . Denote  $\underline{\mathbf{w}} \underline{\mathbf{z}}^T = E$  and compute  $E^2$ . Then compute  $E^3$ . State what pattern for  $E^k$  you predict.

(k) Using this pattern, you can relate the physical size (i.e., that measured with a ruler) of  $C^6$  and  $C^7$  applied to your letter with the number  $s$  defined in part (j). Make such measurements and state what  $s$  is.

**Submission instructions:**

1. Please submit as a single pdf file only pages 2 and 3 (i.e., the next two pages) of this Answer Sheet, as well as all supporting handwritten work. Please do *not* include either this page with submission instructions or/and the Project description itself.
2. There will be a significant number of plots and printouts (of codes) in this project. In order both to organize your work and to save paper/file size, please put any of your printouts and plots into a Word file (see below), convert it to a pdf, append any handwritten work to the same pdf file, and submit only that file. If this is not done, I will **reduce your score by 5 points**.

To include a figure into your main file, I recommend the following steps. Create a .png, .jpeg, or .tiff figure; Insert it into a Word document; Convert the Word document into a pdf. The syntax in Matlab for creating a .png file is: `print -dpng 'foldername/filename'`; see help print for more details. It is similar for other picture formats. Append any handwritten work to it as needed.

3. In addition to the printouts of your codes included in the submitted file, you must also e-mail me these codes. These codes should produce the action required in the assignment when I run them. For each code which you submit as *either* a hard or soft copy *but not both*, I will **reduce your score by 5 points**.
4. The subject line of your e-mail must contain the string MATH\_122. It is case-insensitive; thus, math\_122 or Math\_122 will also work. However, the underscore must be present: e.g., math122 or math 122 will *not* work. Moreover, the subject line must indicate the fact that this is a submission of codes for Project 5. I will **reduce your score by up to 3 points** if this is not done.
5. Your codes must be named according to this convention: p5\_YourName(s)\_exExercise#.m. E.g., my own code for Exercise 2 would be named p5\_tlakoba\_ex2.m.  
(Alternatively, you may submit one code, p5\_YourName(s).m, where codes for the individual exercises are separated by Matlab's %% sectioning command.)  
For each code whose name does not follow this convention, I will **reduce your score by 2 points**.
6. Finally, suggestion and a **strong request**.

Suggestion: Include the command `clear all` at the beginning of your code for Exercise 1.

Strong request: Contrary to what you might have learned in other classes, I ask you **not** to put commands `clc` and `close all` at the beginning of your codes. If I want to clear my command window and close figure windows, I will do so myself, without unsolicited outside "help".

*Answer Sheet is continued on next two pages*



*Continued from previous page*

**Exercise 3** (27 points)

(a) and (b) Attach the code and the plot.

How is the appearance of your final letter in part (a) related to the eigenvalue(s) and eigenvector(s) of  $C$ ? Be as specific about the involved eigenvector(s) and its (their) corresponding eigenvalue(s) as possible.

(c) Read the instructions and fill in the blanks:

Let  $A$  be a matrix with *distinct* eigenvalues and let  $\mathbf{u}$  be an arbitrary vector. If one multiplies  $\mathbf{u}$  by  $A$  \_\_\_\_\_ times, then the result looks similar to the \_\_\_\_\_ of  $A$  that corresponds to the \_\_\_\_\_ eigenvalue.

**Exercise 4** (9 points)

Should the result look different? Make sure to provide an explanation in the attached Word file or on separate pages.

**Bonus** (points per part: (a)→ 7, (b)→ 9, (c)→ 6, (d)→ 12, (e)→ 7, (f)→ 9, (g)→ 12, (h)→ 7, (i)→ 11, (j)→ 8, (k)→ 9)

Credit for each part will be given only if the solution is at least 60% correct and is clearly presented.

Attach your work on separate pages or/and in the Word file.