Lec. 1 – Review

(based mostly on Chap. 5 of Barnett's Calculus)

Reminder of derivative notation:

$$F'(x) = \frac{dF(x)}{dx}$$

E.g., $(x^4)' = 4x^3$

$(3x^4+5)' = 3\cdot 4x^3 + 0$, etc.

1 Integration rules

- If $F(x)$ is such that (s.t.) $F'(x) = f(x)$, then

$$\underbrace{\int f(x)\,dx}_{\text{indefinite } \int \text{ of } f(x)} = \underbrace{F(x)}_{\text{anti-derivative of } f(x)} + C$$

arbitrary const.

Indefinite integrals of basic functions:

$\boxed{R1}$. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

$\boxed{R2}$. $\displaystyle\int e^x dx = e^x + C$

$\boxed{R3}$ $\displaystyle\int \frac{1}{x} dx = \ln|x| + C$, $x \neq 0$

Q: What happens to $\boxed{R1}$ when $n = -1$?

Ex. 1 $\displaystyle\int x^{\overset{n=3}{3}} dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$

Properties of indefinite integral

$\boxed{R4}$. $\int \underset{\substack{\uparrow \\ \text{const}}}{K} \cdot f(x)\,dx = K \cdot \int f(x)\,dx$

$\boxed{R5}$. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Ex. 2a $\int (5x^6 - 7e^x)\,dx \overset{\boxed{R5}}{=} \int 5x^6\,dx - \int 7e^x\,dx$

$\overset{\boxed{R4}}{=} 5\int x^6\,dx - 7\int e^x\,dx \overset{\boxed{R1},\boxed{R2}}{=} 5 \cdot \frac{x^{6+1}}{6+1} - 7e^x + C$

$= \frac{5}{7}x^7 - 7e^x + C.$

Q: why did we write "$+C$" instead of

$5\left(\frac{x^7}{7} + C_1\right) - 7(e^x + C_2)$?

A: If we write that, we can combine the constants:

$= \frac{5}{7}x^7 + \underline{5C_1} - 7e^x - \underline{7C_2} = \frac{5}{7}x^7 - 7e^x + \underbrace{(5C_1 - 7C_2)}_{\text{call this "}C\text{"}}$

Ex. 2b Sometimes the variable can be other than x.

$\int \frac{4}{t}\,dt = 4\int \frac{dt}{t} = 4\ln|\underset{\substack{\uparrow \\ \text{not } x}}{t}| + C$ (not $4\ln|x| + C$).

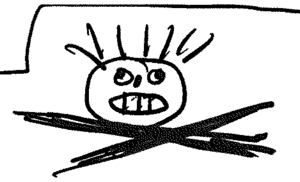
Caution ⚠ : Property $\boxed{R4}$ holds only when $K = \text{const}$.

Namely: $\int \underset{\substack{\uparrow \\ \text{const}}}{K} \cdot f(x)\,dx = K \cdot \int f(x)\,dx$, BUT:

$\int g(x) \cdot f(x) \neq g(x)\int f(x)\,dx$, or $\neq \int g(x)\,dx \int f(x)\,dx$, etc.

(see p. 327)

General Rule of Thumb:
If a formula is not given,
do **not** invent your own!



More Examples on $\boxed{R1}$:

Ex. 3a (negative powers)   Similar in book: Ex. 5.1.3A

$$\int \frac{5}{x^3}\,dx = \int 5\,x^{-3}\,dx = 5\int x^{-3}\,dx = 5\,\frac{x^{-3+1}}{-3+1} + C$$

$n=-3$

$$= \frac{5x^{-2}}{-2} + C = -\frac{5}{2}\cdot\frac{1}{x^2} + C.$$

Ex. 3b (fractional powers)   Similar in book: Ex. 5.1.3B

$$\int 6\sqrt[4]{t^3}\,dt \underset{\text{Appendix 6}}{=} \int 6\,t^{3/4}\,dt = 6\cdot\frac{t^{3/4+1}}{3/4+1} + C$$

n

$$= 6\,\frac{t^{7/4}}{7/4} + C = \frac{24}{7}\,t^{7/4} + C \qquad 6\cdot\frac{1}{7/4} = 6\cdot\frac{4}{7}$$

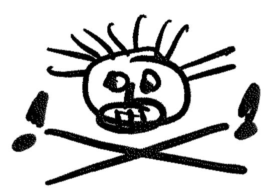
Ex. 3c (simplify fraction before integration); in book Ex. 5.1.3C

$$\int \frac{x^4+5}{x}\,dx \underset{\text{split fraction}}{=} \int\left(\frac{x^4}{x} + \frac{5}{x}\right)dx \underset{\text{Appendix 5}}{=} \int\left(x^{4-1} + \frac{5}{x}\right)dx$$

$$= \int\left(x^3 + \frac{5}{x}\right)dx \underset{\boxed{R5,1,4}}{=} \frac{x^{3+1}}{3+1} + 5\ln|x| + C$$

$$= \frac{x^4}{4} + 5\ln|x| + C$$

Caution ⚠ Cannot do this



$$\underbrace{\int \frac{x}{x^4+5}\,dx = \int\left(\frac{x}{x^4} + \frac{x}{5}\right)dx}_{\text{SUPER WRONG!}}$$

Reason: While $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ ← CORRECT,

$\boxed{\frac{c}{a+b} = \frac{c}{a} + \frac{c}{b} \text{ IS WRONG}}$.

E.g.:
- $\frac{1}{2+3} = \frac{1}{5}$, but

  $\frac{1}{2+3} \neq \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \left(\neq \frac{1}{5}!\right)$
- $\frac{2}{3-4} = \frac{2}{-1} = -2$, but

  $\frac{2}{3-4} \neq \frac{2}{3} - \frac{2}{4} = \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{3} \left(\neq -2!\right)$.

↑ Avoid the above mistake: it will be strongly penalized. ↑

Caution ⚠ Also can NOT do this:

$$\int \frac{x^4+5}{x}\,dx \neq \frac{\int (x^4+5)\,dx}{\int x\,dx}.$$ (see p. 329)

(Again, do not invent your own formulas beyond Rules 1-5.)

Ex. 3d (simplify the product before integrating); book Ex. 5.1.3E

$$\int x^2(x^3+4)\,dx \underset{\substack{\uparrow \\ \text{expand}}}{=} \int (x^2 x^3 + 4x^2)\,dx \underset{\substack{\uparrow \\ \text{Appendix 5}}}{=} \int (x^{2+3} + 4x^2)\,dx$$

$$= \int (x^5 + 4x^2)\,dx = \frac{x^{5+1}}{5+1} + 4\frac{x^3}{3} + C = \frac{x^6}{6} + 4x^3 + C$$

Ex. 3e extension of Ex. 3d

$$\int (x^2+x)(x^3+4)\,dx \underset{\substack{\uparrow \\ \text{expand the integrand}}}{=} \int (x^5+4x^2+x^4+4x)\,dx$$

$(x^2+x)(x^3+4) =$
$x^2(x^3+4) + x(x^3+4) =$
$(x^{2+3}+4x^2) + (x^{3+1}+4x) =$
$x^5+4x^2+x^4+4x$

$$= \frac{x^6}{6} + 4\frac{x^3}{3} + \frac{x^5}{5} + 4\frac{x^2}{2} + C.$$

$\hookrightarrow 2x^2$

② Integration by substitution (a.k.a. u-substitution)

Ex. 4a $\int 2xe^{x^2}\,dx$.

This doesn't look like any of Rules 1-5 or above Examples.

(Remember: Do not invent your own rules.)

Rule of thumb for u-sub:
Call the argument of the most complicated part of the function, u (or maybe some other letter).

Step 1 $u = x^2$

Step 2 Compute du using the rule from Sec. 5.2:

If $y = f(x)$, then $dy = f'(x)\,dx$

Similarly: $u = x^2 \Rightarrow$

$du = (x^2)'\,dx = 2x\,dx$

Step 3 Express everything in terms of u & du (no x!)

$$\int 2xe^{x^2}\,dx = \int e^{\overset{u}{x^2}} \underbrace{2x\,dx}_{du} = \int e^u\,du \overset{\boxed{R2}}{=} e^u + C$$

Step 4: Go back from u to x:

$$= e^{(x^2)} + C.$$

Ex. 4b $\int e^{-5t}\,dt$

Step 1 $u = -5t$

Step 2 $du = (-5t)'\,dt = -5\,dt$, $\boxed{\Rightarrow dt = \frac{du}{-5}}$

Step 3 $\int e^{-5t}\,\underbrace{dt}_{-\frac{du}{5}} = \int e^{u}\left(-\frac{1}{5}\cdot du\right)$

$u = -5t$

$= -\frac{1}{5}\int e^{u}du = -\frac{1}{5}e^{u} + C$

Step 4 $= -\frac{1}{5}e^{-5t} + C$

Ex. 5 $\displaystyle\int \frac{2\,du}{3u-4}$ Letter "u" already used; so use another letter, say, **w**.

Step 1 $w = 3u-4$

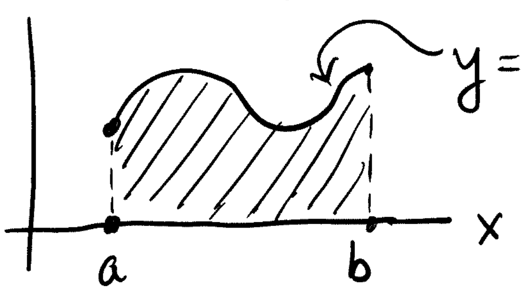
Step 2 $dw = (3u-4)'du = 3du, \Rightarrow du = \frac{1}{3}dw$

Step 3 $\displaystyle\int \frac{2\,du}{3u-4} = \int \frac{2\cdot\frac{1}{3}dw}{w} = \frac{2}{3}\int\frac{dw}{w} = \frac{2}{3}\ln|w| + C$

Step 4 $= \frac{2}{3}\ln|3u-4| + C$

③ Definite integrals & Fundamental Theorem of Calculus

a Meaning of definite integral

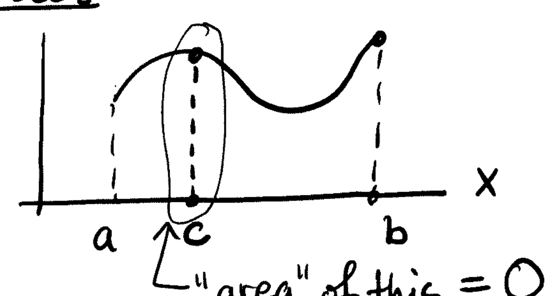


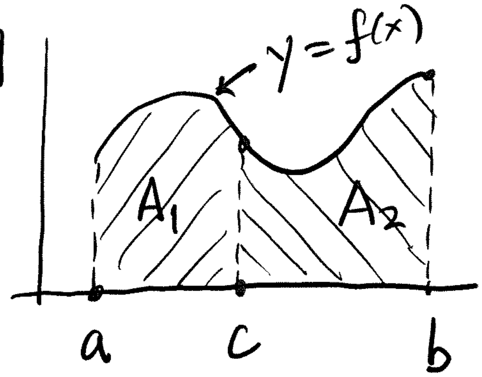
$\displaystyle\int_a^b f(x) =$ area under graph $y = f(x)$ over $a \le x \le b$ when $f(x) \ge 0$.

Rules for definite integrals

D1 $\displaystyle\int_c^c f(x)\,dx = 0$ for any c and any $f(x)$

same





$$\left(\int_a^c + \int_c^b\right) f(x)\,dx = \int_a^b f(x)\,dx$$

(Total area over $[a, b]$ $= A_1 + A_2$.)

(6) Fundamental Theorem of Calculus (FTC)

- If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$
- In other words: $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Note: The arbitrary constant C in $F(x)$ does not matter for definite integrals.

Ex. 6a Similar Example in book: Ex. 5.5.5

Suppose that a population $P(t)$ of some bacteria is decreasing at the rate $P'(t) = -7e^{-0.5t}$. Find the change of the population between $t=2$ & $t=10$.

Note: The word **decrease** always means that the rate of change is **negative**. For example, if a problem is worded: " the population is decreasing at the rate $7e^{-0.5t}$ ", where you note no minus in front of the value, you should have still assumed that $P'(t) = -7e^{-0.5t}$. Note the **minus sign**. It is brought about by the word "**decrease**".

Solution: 1) $P(10) - P(2) \overset{\boxed{FTC}}{=} \int_2^{10} P'(t)\,dt =$

$$= \int_2^{7} -7e^{-0.5t}\,dt = -7\int_2^{10} e^{-0.5t}\,dt.$$

2) Use the u-sub as in Ex. 4b:

$$u = -0.5t$$

$$du = (-0.5t)'dt = -0.5\,dt \Rightarrow dt = \frac{du}{-0.5} = -2du$$

$u(t=2) = -0.5\cdot 2 = -1 \leftarrow$ new lower bound

$u(t=10) = -0.5\cdot 10 = -5 \leftarrow$ new upper bound

above $\int = -7\int_{-1}^{-5} e^u \cdot (-2)\,du = (-7)(-2)\int_{-1}^{-5} e^u\,du$

$= 14\cdot e^u\Big|_{-1}^{-5} = 14\left(e^{-5} - e^{-1}\right) \approx -5.06$

(The change is < 0 because the population decreases.)

Ex. 6b In Ex. 6a, suppose that the population at $t=2$ is known to be $14/e$. At what time T will it become $1/3$ of $P(2)$?

Solution:

1) $P(T) = \underbrace{P(2)}_{\text{given value at } t=2} + \underbrace{(P(T) - P(2))}_{\text{change between } t=2 \text{ \& } t=T.}$

2) Let us first find a formula for $P(T) - P(2)$. All steps of Ex. 6a are the same except that $u(T) = -0.5T$. This number replaces $u(10) = -5$.

Then:

$$P(T) - P(2) = 14\left(e^{-0.5T} - e^{-1}\right).$$

3) Find $P(T)$ from 1) & 2):

$$P(T) = \underbrace{\frac{14}{e}}_{P(2)} + \underbrace{14\left(e^{-0.5T} - e^{-1}\right)}_{P(T)-P(2)}.$$

Recall that $\frac{1}{e} = e^{-1}$, $\Rightarrow \frac{14}{e} = 14e^{-1}$.
These terms cancel above. Then

$$P(T) = 14e^{-0.5T}$$

4) Use the condition $P(T) = \frac{1}{3}P(2)$:

$$\cancel{14}e^{-0.5T} = \frac{1}{3}\cdot\cancel{14}e^{-1}$$

$$e^{-0.5T} = \frac{1}{3}e^{-1}$$

Recall from Sec. 1.6 :

$$\ln(e^a) = a$$

then $\underline{e^a = b} \Rightarrow \ln(e^a) = \ln b \Rightarrow \underline{a = \ln b}$.

Applying this to our equation:

$$\ln\left(e^{-0.5T}\right) = \ln\left(\tfrac{1}{3}e^{-1}\right) \Rightarrow -0.5T = \underbrace{\ln\left(\tfrac{1}{3}e^{-1}\right)}_{\approx -2.10}$$

$$T \approx \frac{\cancel{-}2.10}{\cancel{-}0.5} = \underline{4.20.}$$ Answer.

Ex. 7 Similar in book: Ex. 5.5. {3,4}

Find $\displaystyle\int_{-3}^{-2} \frac{dx}{\sqrt{4-x}}$.

Sol'n: Realize that need a u-sub

1-10

Step 1: $u = 4 - x$

Step 2: $du = (4-x)'\,dx = -1 \cdot dx \Rightarrow dx = -du$

Step 3 for definite integrals:

New lower bound $= u(-3) = 4-(-3) = 7$

New upper bound $= u(-2) = 4-(-2) = 6$

Step 4 for definite integrals:

$$\int_{-3}^{-2} \frac{dx}{\sqrt{4-x}} = \int_{7 \leftarrow u(-3)}^{6 \leftarrow u(-2)} \frac{-du \leftarrow dx}{\sqrt{u}} = -\int_7^6 u^{-1/2}\,du$$

$$\overset{\boxed{R1}}{=} -\frac{u^{-1/2+1}}{\underbrace{-1/2+1}_{+1/2}}\Bigg|_7^6 = -2u^{1/2}\Big|_7^6 =$$

$$= -2\left(6^{1/2} - 7^{1/2}\right) = 2\left(\sqrt{7} - \sqrt{6}\right)$$

flip sign, change order of terms