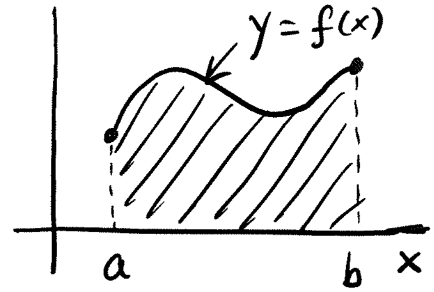
Lec. 2 : Area between curves

0 Overview

From Lec. 1, topic 3, recall that

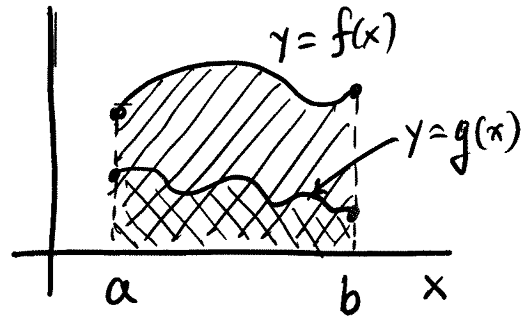
$$\int_a^b f(x)\,dx = \text{area under } y = f(x) \text{ over interval } a \le x \le b \text{ (when } f(x) \ge 0\text{)}.$$



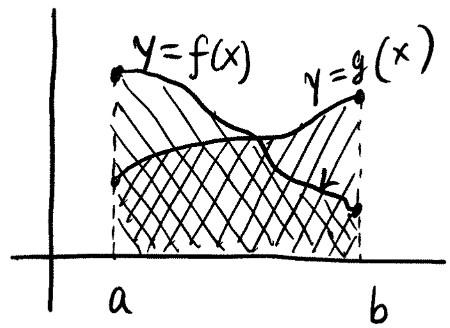
In this Lecture we consider two extensions:

We will discuss how the area between the curves can be found and what it represents.
In the next Lecture, we will consider applications of the area concept.

1



2



In both cases, the area is given by:

(Area under f) − (Area under g) =

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \boxed{\int_a^b \big(f(x) - g(x)\big)\,dx}$$

Formula to be used in this Lecture.

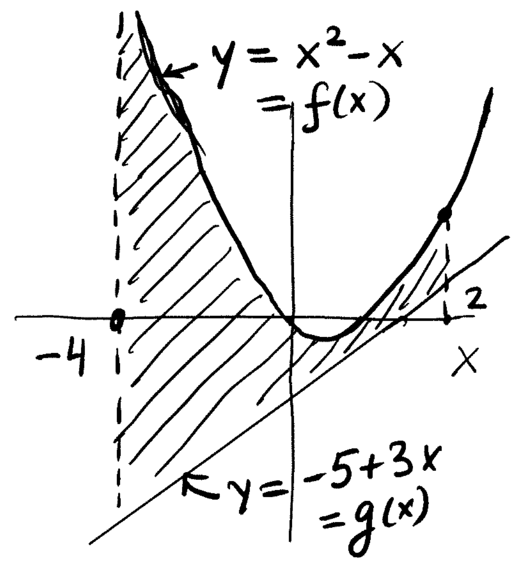
However, its interpretations are different in Cases 1 and 2.

Moreover, the treatment of Case 2 is different from that in the book.
You are expected to follow the treatment of this Lecture, not that of the book.

① Case 1: $f(x) \geq g(x)$ on $a \leq x \leq b$

Ex. 1 Similar in book = Ex. 6.1.3

Find the area between $y=-5+3x$ and $y=x^2-x$ over interval $-4 \leq x \leq 2$.



Sol'n:

1) Sketch (using a calculator) the two curves to verify if one is strictly above the other. Name top curve $y=f(x)$, bottom curve $y=g(x)$.

2) By the general Formula,

$$\text{Area} = \int_a^b (f(x)-g(x))\,dx = \int_{-4}^{2} \left((x^2-x)-(-5+3x)\right)dx$$

$$= \int_{-4}^{2} (x^2 \underbrace{-x +5 -3x}_{\text{combine}})\,dx = \int_{-4}^{2} (x^2-4x+5)\,dx =$$

use $\int x^n dx$

$$\left(\frac{x^3}{3} - 4\frac{x^2}{2} + 5x\right)\Bigg|_{-4}^{2} = \left(\frac{2^3}{3} - 2\cdot 2^2 + 5\cdot 2\right) - \left(\frac{(-4)^3}{3} - 2\cdot(-4)^2 + 5\cdot(-4)\right)$$

$$= \underbrace{\left(\frac{8}{3} \underbrace{-8+10}_{+2}\right)}_{\frac{8}{3}+2=\frac{14}{3}} - \underbrace{\left(-\frac{64}{3} \underbrace{-32-20}_{-52}\right)}_{-\frac{64}{3}-52=\frac{-64-156}{3}=-\frac{220}{3}} = \frac{14}{3} - \left(-\frac{220}{3}\right) = \frac{14}{3} + \frac{220}{3}$$

$$= \boxed{\frac{234}{3}}$$

Answer.

Notes: ① No need to simplify.

② Please don't use decimals unless asked to do so.

Ex. 2 Similar in book = Ex. 6.1.4

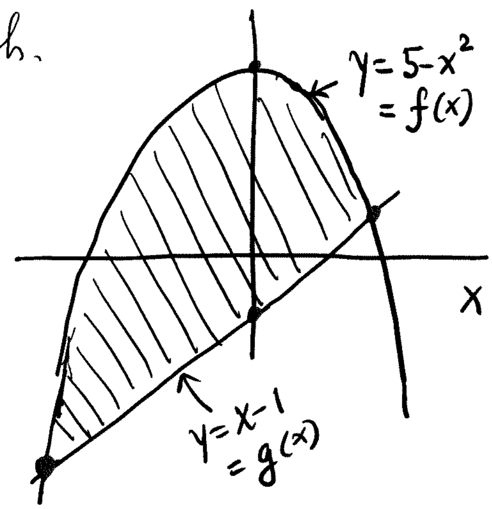
Sometimes bounds are not given; then you need to find them.

Find the area bounded by $y = 5 - x^2$ and $y = x - 1$.

Sol'n:

1) As before, begin by making a sketch. The bounds for the integral come from solving

$$f(x) = g(x).$$



$$5 - x^2 = x - 1$$

Move terms so that the coefficient of x^2 is positive:

$$x^2 + x - 1 - 5 = 0$$

$$\underset{a=1}{x^2} + \underset{b=1}{x} \underbrace{-6}_{c=-6} = 0$$

factor this quadratic equation either by inspection or by the quadratic formula (**must review Appendix 7 and know the formula**)

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{1-(-24)}}{2} = \frac{-1 \pm \sqrt{25}}{2}$$

$$= \frac{-1 \pm 5}{2} = \left[\begin{array}{l} \frac{-1+5}{2} = 2 \\ \frac{-1-5}{2} = -3 \end{array}\right.$$

So $\boxed{\begin{array}{l} x_1 = -3 \\ x_2 = 2 \end{array}}$

If you used inspection, you'd get

$$x^2 + x - 6 = (x+3)(x-2) \quad \Big| \quad = x^2 \underbrace{-2x + 3x}_{x} - 6 \ \checkmark$$

$$= (x - \underbrace{(-3)}_{x_1})(x - \underbrace{2}_{x_2})$$

On quiz/test, can use either the quadratic formula or inspection. For the latter, must show steps of expanding and verifying the expression.

2) Set up the formula for the area:

$$\text{Area} = \int_{x_1}^{x_2} (\underbrace{f(x)}_{\text{on top}} - \underbrace{g(x)}_{\text{on bottom}})\,dx = \int_{-3}^{2} \Big(\underbrace{(5-x^2)-(x-1)}_{5-x^2-x+1 \,=\, 6-x^2-x}\Big)\,dx$$

3) Evaluate the integral:

$$\int_{-3}^{2} (6-x^2-x)\,dx = \left(6x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{-3}^{2} =$$

$$= \left(6\cdot 2 - \frac{2^3}{3} - \frac{2^2}{2}\right) - \left(6\cdot(-3) - \frac{(-3)^3}{3} - \frac{(-3)^2}{2}\right)$$

$$10 - \frac{8}{3} = \frac{22}{3} \qquad \underbrace{-18-(-9)}_{-9} - \frac{9}{2} = \frac{-27}{2}$$

$$= \frac{22}{3} - \left(\frac{-27}{2}\right) = \frac{22}{3} + \frac{27}{2} = \frac{22\cdot 2 + 27\cdot 3}{6} = \frac{44+81}{6}$$

$$= \boxed{\frac{125}{6}} \leftarrow \text{Answer.}$$

Again: Please do not simplify and don't use decimals unless explicitly asked to do so.

Aside on Ex. 2 = Ex. 6.1.6 in book

(This is done JFYI. Problems like this will NOT be assigned.)

Find the area between $y = e^{-x^2}$ and $y = x^2$.

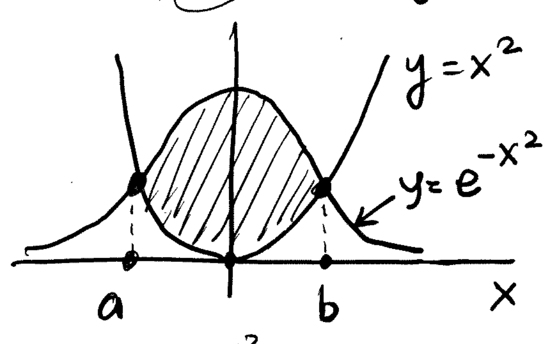
Solution:

1) Sketch with a calculator.

Find the intersection points also with a calculator.

Here: $a \approx -0.753$

$b \approx 0.753$



$y = e^{-x^2}$ is a bell-shaped curve important in problems with probability.

2) Set up the formula for the area:

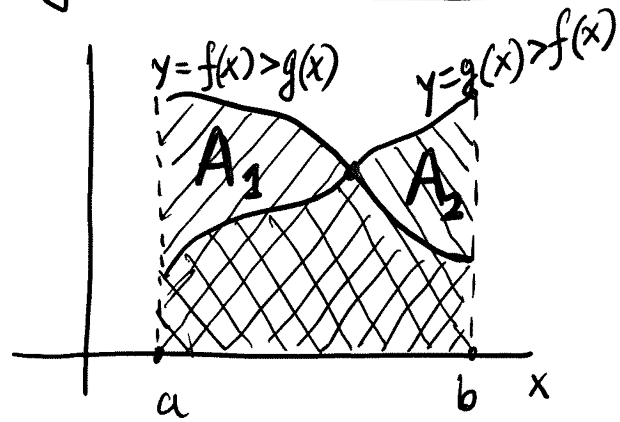
$$A = \int_{-0.753}^{0.753} \left(\underbrace{e^{-x^2}}_{\text{on top}} - \underbrace{x^2}_{\text{on bottom}} \right) dx$$

3) Evaluate using a calculator (some command for numerical integration):

$$= 0.979\ldots$$

② Case 2: Graphs of f and g intersect on $[a, b]$

Here we deviate from the book, and you are expected to **follow the approach of the Lecture, not that of the book.**



The book emphasizes the computation of $A_1 + A_2$ where both A_1 and A_2 are treated as positive.

On the contrary, **when we compute**

$$A_1 + A_2 = \int_a^b (f(x) - g(x))\, dx,$$

we will treat A_1 as positive and A_2 as negative.
$(f > g)$ $(f < g)$

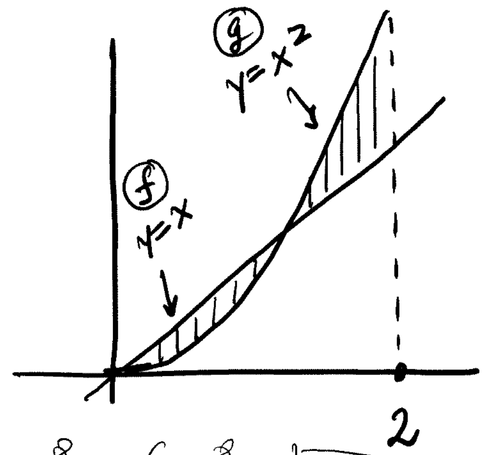
Then

$$\int_a^b (f(x) - g(x))\, dx = \underbrace{\int_a^b f(x)\, dx}_{\text{area under } f} - \underbrace{\int_a^b g(x)\, dx}_{\text{area under } g}$$

tells us how much the area under f is greater (or less) than the area under g.

Ex. 3 Compare the areas under $y = x$ and $y = x^2$ over $0 \le x \le 2$.

Solution: 1) Sketch; label f & g



2) $A_f - A_g = \int_0^2 x\,dx - \int_0^2 x^2\,dx$

$= \int_0^2 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2$

$= \left(\frac{2^2}{2} - \frac{2^3}{3}\right) - (0-0) = \left(\frac{4}{2} - \frac{8}{3}\right) = 2 - \frac{8}{3} = \frac{6}{3} - \frac{8}{3} = \boxed{-2/3}$

Answer

Q: Which area is greater?