Lec. 3: Area between curves - Applications

① Income stream with compounded interest

In Sec. 3.1 you learned the formula for a continuously compounded interest on an investment:

$$\boxed{A = P \cdot e^{rt}}$$

A: value of investment at time t

P: principal (= initial investment)

r: interest rate (compounded continuously)

Ex. 1(a) Similar in book = Text on p. 402

Suppose you deposit \$2000 every year to an account that pays 8% annual rate (compounded continuously). What will be your **accumulated interest** from that account at the end of the 10th year?

(Here rate $r = 0.08$ if t is measured in years.)

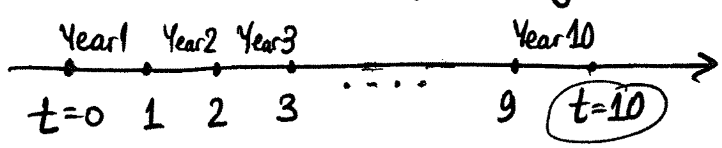
Sol'n: Year 1 Deposit \$2000 at $t=0$.

Amount at the end of Year 10 (i.e. at $t=10$), called **Future Value** (FV):

$$A = 2000 \cdot e^{0.08 \cdot 10}$$

Important Note:

In all such problems, draw the timeline; othewise your t can end up being off by 1.



Year 2 Deposit \$2000 at $t=1$.

FV at end of Year 10 $= 2000 \cdot e^{0.08 \cdot (10-1)}$ ← interest accumulates for 9 years

Year 3 Deposit \$2000 at $t=2$.

FV at end of Year 10 $= 2000 \cdot e^{0.08 \cdot \underbrace{(10-2)}}$

accumulates for 8 years

⋮

Year 10 Deposit \$2000 at $t=9$ (see the timeline!)

FV at end of Year 10 $= 2000 \cdot e^{0.08 \cdot \underbrace{(10-9)}}$

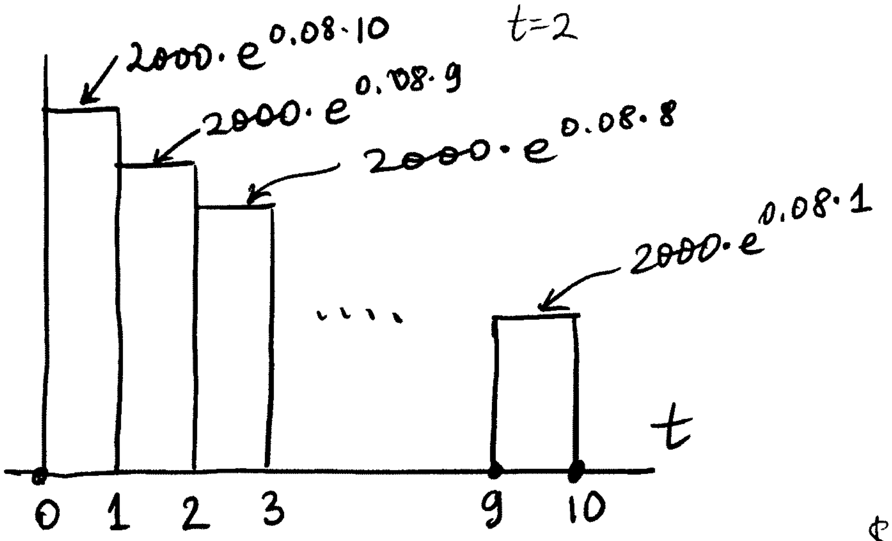
accumulates for only 1 year.

$$\text{Total FV} = 2000 \cdot e^{0.08 \cdot (10 - \underset{t=0}{0})} + 2000 \cdot e^{0.08(10 - \underset{t=1}{1})}$$

$$+ 2000 \cdot e^{0.08 \cdot (10 - \underset{t=2}{2})} + \dots + 2000 \cdot e^{0.08 \cdot (10 - \underset{t=9}{9})}$$

$\approx \boxed{\$31,880}$

FV at end of Year 10.



Total FV = Combined area of rectangles

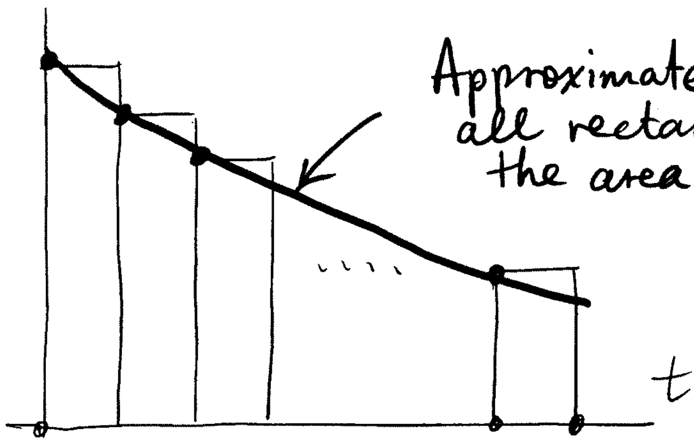
Accumulated interest

=

$\$31,880 - \underbrace{\$2000 \cdot 10}_{\text{total investment}}$

$= \$11,880$.

Ex. 1(b) Approximate this using a continuous deposit model.



Approximate area under all rectangles by the area under continuous curve

$$y = 2000 \cdot e^{0.08 \cdot (10-t)}$$

Observations

- The continuous approximation is seen to underestimate the value found with the once-a-year deposit model.
- How big is the error?

Area under the continuous curve $= \int_0^{10} 2000 \cdot e^{0.08(10-t)}\,dt$

$= 2000 \cdot \int_0^{10} e^{0.08(10-t)}\,dt \underset{\uparrow}{=} 2000 \int_0^{10} e^{\overset{0.8}{0.08\cdot 10} - 0.08t}\,dt$

expand the exponent

$\underset{\uparrow}{=} 2000 \cdot \int_0^{10} e^{0.8} \cdot e^{-0.08t}\,dt = 2000 \cdot e^{0.8} \int_0^{10} e^{-0.08t}\,dt.$

use $e^{m-n} = e^m \cdot e^{-n}$
(Appendix 5)

This integral is done as in Ex. 4b in Lec. 1:

$\int_0^{10} e^{-0.08t}\,dt$

$= \int_0^{-0.8} e^u \left(-\frac{du}{0.08}\right)$

$= -\frac{1}{0.08} \int_0^{-0.8} e^u\,du$

$= -\frac{1}{0.08} e^u \Big|_0^{-0.8} =$

$u = -0.08t$
$du = (-0.08t)'dt = -0.08\,dt$
$dt = -du/0.08$
New lower bound:
$u(0) = -0.08 \cdot 0 = 0$
New upper bound:
$u(10) = -0.08 \cdot 10 = -0.8$

$-\frac{1}{0.08}\left(e^{-0.8} - \cancel{e^0}^{\,1}\right) = +\frac{1-e^{-0.8}}{0.08}.$

Going back to our original expression $=$

$2000 \cdot e^{0.8} \cdot \frac{(1-e^{-0.8})}{0.08} \underset{\uparrow}{=} 2000\,\frac{e^{0.8}-1}{0.08}.$

distribute $e^{0.8}$

1/27/23

Putting this in a calculator gives ≈ \$30,639

Total FV using continuous model

<u>Once-per-year model</u> | <u>Continuous model</u>

Total FV ≈ 31,880 | Total FV ≈ \$30,640

error = |31,880 − 30,640| = 1,240,
or about 4% (1240/31,000 ≈ 0.04)

Accumulated interest = 31,880 − 20,000 = 11,880

Accumulated interest = 30,640 − 20,000 = 10,640

error = 1,240, or about 11% ($\frac{1240}{31,000} \approx 0.11$)

Not great, but can be "good enough"
(and there is a nice formula).

- At home I ask you to trace the steps of the above calculation for the <u>general</u> case, with:

$$2000 \to P$$
$$0.08 \to r$$
$$10 \to T$$

and obtain **the general formula:**

$$\boxed{\int_0^T P e^{r(T-t)}\,dt = P e^{rT} \cdot \frac{1-e^{-rT}}{r} = P \cdot \frac{e^{rT}-1}{r}}$$

<u>Ex. 2</u> Similar to Ex. 6.2.4 in **book**.
Redo the previous problem if you are <u>changing the deposit with time</u> as $P(t) = 2000 \cdot e^{-0.03t}$.
Approximate the answer using a continuous model.

Sol'n : 1) We begin by following the setup of the discrete (once-per-year deposit) of Ex. 1(a). It is modified to account for a changing amount of the deposits:

Total FV at end of Year 10 =

$$\underbrace{2000 \cdot e^{-0.03 \cdot 0} \cdot e^{0.08 \cdot (10-0)}}_{\text{from deposit at start of Year 1}} \quad t=0 \quad + \underbrace{2000 \cdot e^{-0.03 \cdot 1} \cdot e^{0.08(10-1)}}_{\text{from deposit at start of Year 2}} \quad t=1$$

$$+ \underbrace{2000 \cdot e^{-0.03 \cdot 2} \cdot e^{0.08(10-2)}}_{\text{from deposit at start of Year 3}} \quad t=2 \quad + \ldots \approx$$

See next Step

2) Approximate this by the continuous model, as in Ex. 1(b)

$$\approx \int_0^{10} 2000 \cdot e^{-0.03t} \cdot e^{0.08(10-t)}\, dt$$

$$= 2000 \cdot \int_0^{10} e^{-0.03t} \cdot e^{0.08 \cdot 10} \cdot e^{-0.08t}\, dt$$

combine using $e^m \cdot e^n = e^{m+n}$

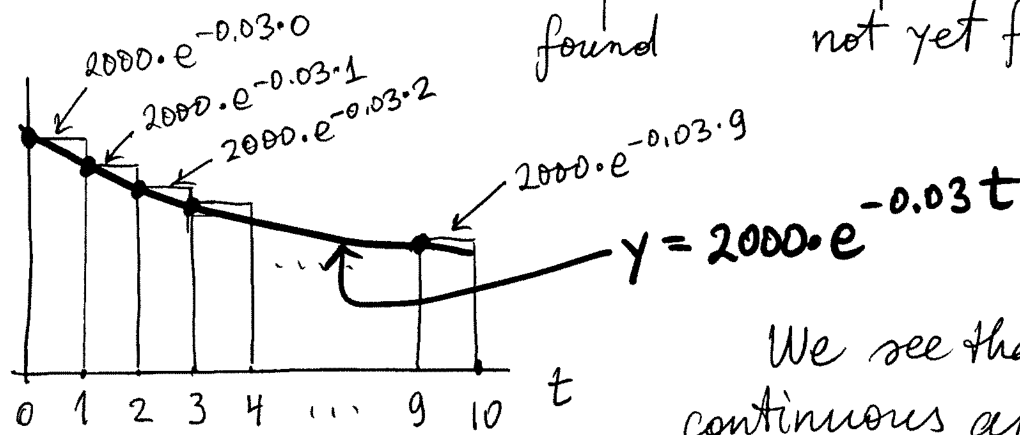
$$= 2000 \cdot e^{0.8} \cdot \int_0^{10} e^{-\underbrace{(0.03+0.08)}_{0.11}t}\, dt$$

$$= 2000 \cdot e^{0.8} \cdot \frac{1 - e^{-0.11 \cdot 10}}{0.11} = \boxed{26{,}995}$$

following the steps of the derivation of the "general formula" on p. 3-4

FV at $t = 10$

3) $\begin{pmatrix}\text{Accumulated}\\ \text{interest}\end{pmatrix} = \underset{\substack{\uparrow\\ \text{found}}}{FV} - \underset{\substack{\uparrow\\ \text{not yet found}}}{\begin{pmatrix}\text{Total}\\ \text{income}\end{pmatrix}}$



We see that the continuous approximation underestimates the exact answer (since the thick curve bounds a smaller area than all of the rectangles). However, we will proceed by using the continuous approximation so as to, as before, get a simple formula. So, we approximate:

Total income =

$$2000\cdot e^{-0.03\cdot 0} + 2000\cdot e^{-0.03\cdot 1} + \dots 2000\cdot e^{-0.03\cdot 9}$$

$$\approx \int_0^{10} 2000\cdot e^{-0.03t}\,dt = 2000\cdot \int_0^{10} e^{-0.03t}\,dt$$

$$\underset{\substack{\uparrow\\ \text{using the } \textbf{general formula}\\ \textbf{from Ex. 1(b)}}}{=} 2000\cdot \frac{1-e^{-0.03\cdot 10}}{0.03} = 17{,}279.$$

$$\begin{pmatrix}\text{Accumulated}\\ \text{interest}\end{pmatrix} = \underset{\substack{\uparrow\\ \text{from Step 2}}}{FV} - \begin{matrix}\text{Total}\\ \text{income}\end{matrix} = 26{,}995 - 17{,}279 = \underline{\underline{9{,}716}}.$$

<u>Q</u>: Why do you think the interest is only ~10% less than in Ex. 1b, even though the total income (= total deposits) are ~25% less than in Ex. 1b?

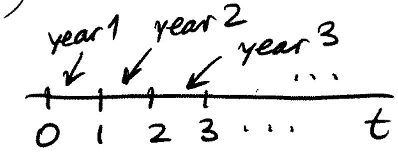
A useful By-Product Results:

1 Suppose a continuos income stream has the rate $f(t)$. (In Ex. 2, $f(t) = 2000 \cdot e^{-0.03t}$.) Then the income accumulated from $t=a$ to $t=b$ is:

$$\text{Total income} = \int_a^b f(t)\,dt$$

(In Ex. 2: $t=0$ = beginning of Year 1; $t=10$ = end of Year 10.)

Reminder: Always sketch a time line:



2 If a continuous income stream with a rate $f(t)$ accumulates interest at a rate r, then the FV of the income at $t = T$ is:

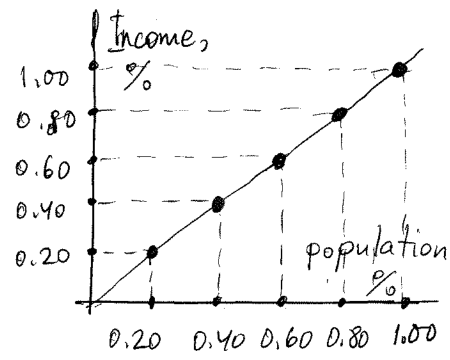
$$FV(T) = \int_0^T f(t)\, e^{r(T-t)}\, dt$$

② Income distribution in a society; income inequality
(pp. 392-394 in textbook)

Suppose that all people/households in a country have the same income. Then:

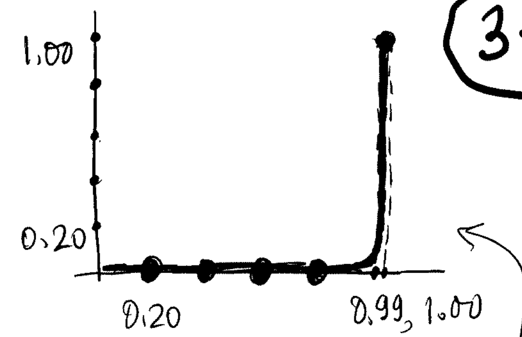
10% of the population earns 10% of the country's total income
20% of the population earns 20% of the country's total income
⋮
50% of the population earns 50% of the country's total income
⋮ etc

If we plot the percentage of country's total income earned by the respective percentage of the population, we'll get a straight line $y = x$.



Thus, this line represents **absolute equality of income distribution**.

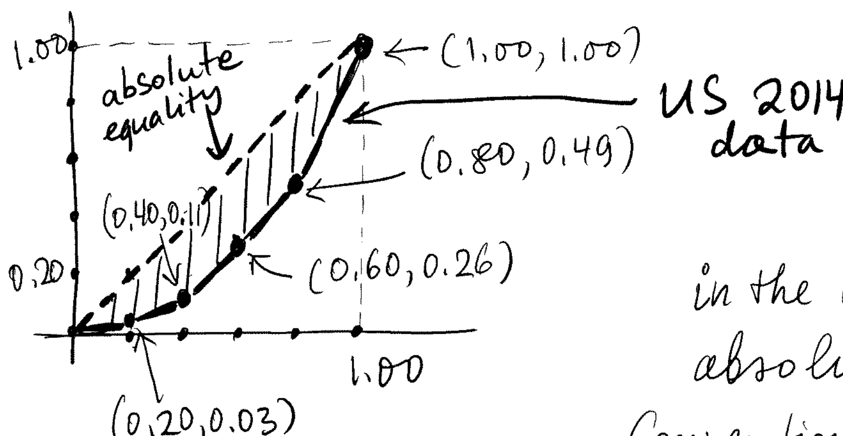
In the opposite extreme, when 99% of the population earn nothing, and all income in concentrated in the hands of 1% of the population, the curve will look like this.



This curve represents

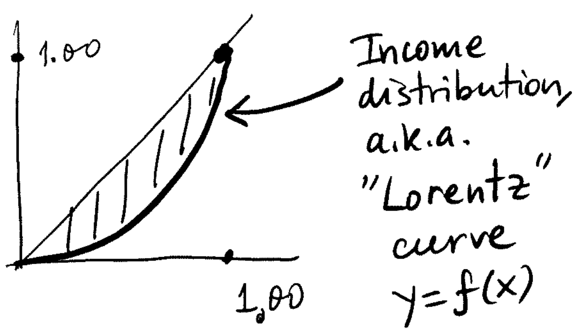
absolute inequality of income distribution.

In reality, the actual income distribution curve is somewhere between these two extremes. ~~For the~~ 2014 income distribution in the US (p. 393 of ~~book~~):



The shaded area shows how far the income distribution in the US in 2014 was from absolute equality.

Convention: normalize (= divide) this area by the area under the abs. equality line $y = x$ (which is $1/2$, by inspection or integration).



Gini index of income concentration:

$$\frac{\int_0^1 (x - f(x))\,dx}{(1/2)} = 2\int_0^1 (x - \underbrace{f(x)}_{\text{Lorentz curve.}})\,dx$$

Gini Index $= 0 \Rightarrow$ abs. equality

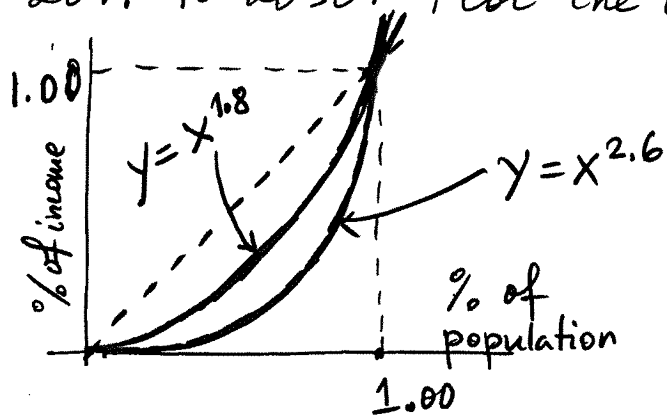
Gini Index $= 1 \Rightarrow$ abs. inequality

The Lorenz curve is usually approximated as $y = x^{a}$ ← some const > 1.

Ex. 3 (= Ex. 6.1.8 in textbook) 3-9

The Lorentz curve for the income distribution in 2017 in some country is $y = x^{2.6}$. Economists predict that by 2030, the Lorentz curve in this country will be approximated by $y = x^{1.8}$. Find the corresponding Gini indices and interpret the results.

Sol'n: 0) We can first qualitatively predict if the income inequality is going to increase or decrease from 2017 to 2030. Plot the curves:



Based on the graph, we can predict that the Gini index (or, equivalently, the area of the crescent) is smaller for $y = x^{1.8}$ than for $y = x^{2.6}$, and so the income inequality is expected to decrease from 2017 to 2030.

1) Now let's find the actual values of the Gini index.

2017: $G.I. = 2\int_0^1 (x - f(x))\,dx = 2\int_0^1 (x - x^{2.6})\,dx =$

$2 \cdot \left(\frac{x^2}{2} - \frac{x^{2.6+1}}{2.6+1}\right)\Big|_0^1 = 2 \cdot \left(\frac{1}{2} - \frac{1}{3.6}\right) - 0 \approx 0.44$

2030: $G.I. = 2\int_0^1 (x - x^{1.8})\,dx = 2\left(\frac{x^2}{2} - \frac{x^{1.8+1}}{1.8+1}\right)\Big|_0^1$

$= 2 \cdot \left(\frac{1}{2} - \frac{1}{2.6}\right) \approx 0.29$

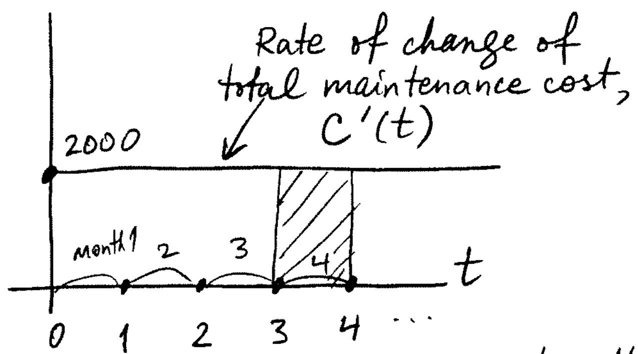
Indeed, $G.I._{2030} < G.I._{2017}$, so income inequality is expected to decrease by 2030.

② Useful life of a product

When a product is introduced to market, it generates Revenue at some rate $R'(t)$ but also has some maintenance Cost per unit time, $C'(t)$. (It is the rate at which the total maintenance cost accumulates, just like $R'(t)$ is the rate at which the total revenue accumulates.)

Ex. 4 (based on Ex. 5.5.6 in textbook)

Preliminaries



A video game is installed in an arcade. It costs \$2000/month to maintain/support this game. So:

$$\underbrace{\frac{\overbrace{\Delta C}^{\text{maintenance cost per month}}}{\Delta t}}_{\text{1 month } \{\Delta t} \approx \frac{dC}{dt} = C'(t) = 2{,}000.$$

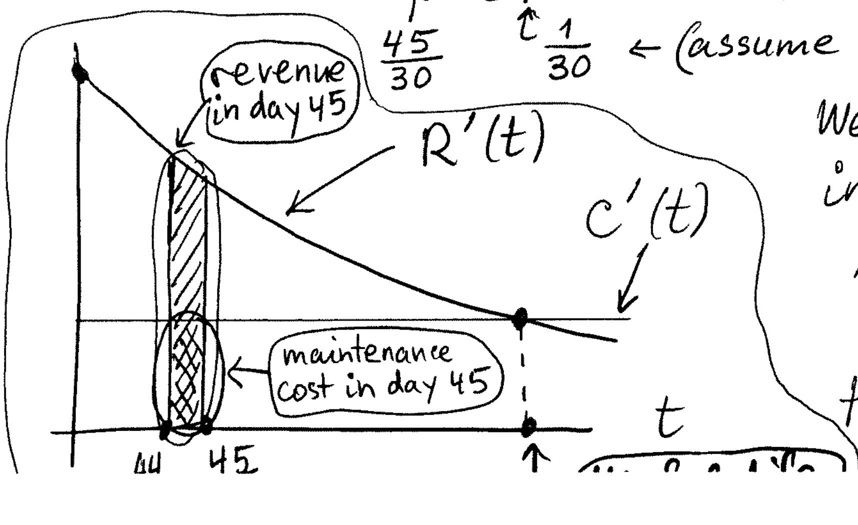
This game generates revenue at a rate $R'(t) = 9{,}000 \cdot e^{-0.5t}$.

time (in months) since the installation

E.g., during its 45th day, it will generate a revenue:

$$\Delta R \approx R'(\underset{\frac{45}{30}}{t})\, \underbrace{\Delta t}_{\frac{1}{30}} = 9000 \cdot \left(e^{-0.5 \cdot \frac{45}{30}}\right) \cdot \frac{1}{30}$$

← (assume 30 days in a month)



We can plot $R'(t)$ and $C'(t)$ in the same graph.

As long as

$$R'(t) > C'(t),$$

the game generates **profit**.

The profit is generated at the rate

$$(R'(t) - C'(t)) = P'(t) \leftarrow \text{rate of profit generation}$$

Therefore :

- The profit accumulated between times t_1 & t_2 is:

$$\begin{pmatrix}\text{Profit} \\ \text{from } t_1 \text{ to } t_2\end{pmatrix} = \int_{t_1}^{t_2} P'(t)\,dt = \int_{t_1}^{t_2} (R'(t) - C'(t))\,dt$$

- Profit increases (= product makes money) when $P'(t) > 0$, or $R'(t) - C'(t) > 0 \Rightarrow R'(t) > C'(t)$.

- Profit decreases (= product loses money) when $P'(t) < 0$, or $R'(t) - C'(t) < 0$, $\Rightarrow R'(t) < C'(t)$.

- **The time t when $\boxed{R'(t) = C'(t)}$ is called the Useful life (U.L.) of the product.**

Questions :

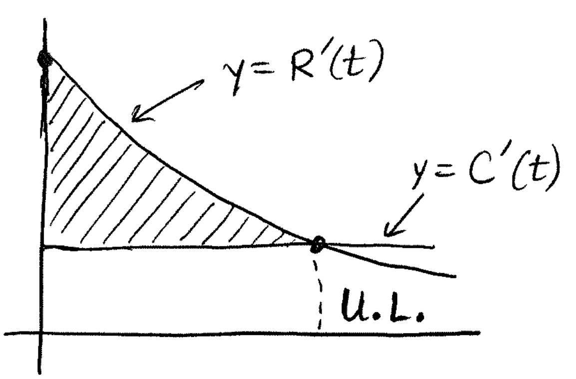
(a) What is the U.L. of the game in this example?

Sol'n: $R'(t) = C'(t) \Rightarrow 9000 \cdot e^{-0.5t} = 2000 \Rightarrow$

$e^{-0.5t} = \frac{2000}{9000} = \frac{2}{9} \Rightarrow -0.5t = \ln\frac{2}{9} \Rightarrow$ U.L. $= -2\ln\frac{2}{9} \approx 3.01 \approx 3$ moths.

(See Ex. 6b in Lec. 1)

(b) What is the total profit generated by this game during its U.L.? (Assume UL=3.)



Graphically, it is the area bounded by $y = R'(t)$ and $y = C'(t)$ (Lec. 2.).

Based on Lec. 2:

$$\begin{pmatrix}\text{Total profit}\\ \text{during U.L.}\end{pmatrix} = P(\underset{\text{U.L.}}{3}) - \underbrace{P(0)}_{\substack{\text{can set}\\ =0}} = \int_0^3 P'(t)\,dt =$$

$$= \int_0^3 (\underbrace{9000e^{-0.5t}}_{R'(t)} - \underbrace{2000}_{C'(t)})\,dt$$

$$= \left[9000\cdot\left(-\frac{1}{0.5}\right)e^{-0.5t} - 2000t\right]_0^3$$

$$= \left[-\frac{9000}{0.5}\left(e^{-0.5\cdot 3} - \underbrace{e^{-0.5\cdot 0}}_{1}\right) - 2000\cdot(3-0)\right]$$

$$\approx \$7,984 \approx \$8,000.$$

Useful formula (see Ex. 4b, 6a / Lec. 1 and Ex. 1b / Lec. 3):

$$\boxed{\int e^{at}\,dt = \frac{1}{a}e^{at} + C}$$

when $a = \text{const}$

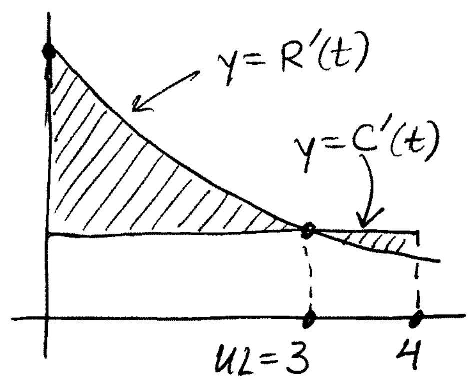
(c) How much money will the owner of the arcade lose by keeping this game for 1 extra month?

Sol'n:

Method 1: Compute $(P(4) - P(0))$

$= \begin{pmatrix}\text{area between } R' \,\&\, C'\\ \text{for } 0 \le t \le 4\end{pmatrix}$

and compare with $(P(3) - P(0))$.

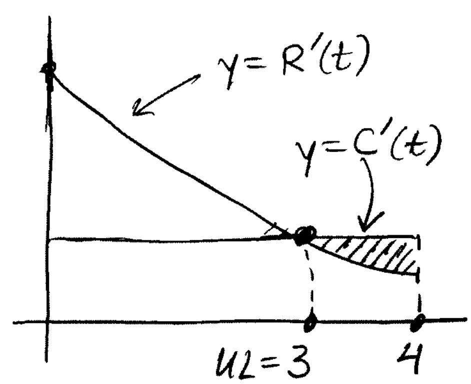


Method 2: Compute $(P(4) - P(3))$:

$$P(4) - P(3) = \int_3^4 (9000e^{-0.5t} - 2000)\,dt =$$

$$\left[-\frac{9000}{0.5}e^{-0.5t} - 2000\cdot t\right]_3^4$$

$$\approx 1580 - 2000 = -420.$$



Answer: Owner will lose $420.