Lec. 4 : Integration by parts

① Background

a Preview:

In this lecture, we'll learn how to evaluate these integrals:

- $\int x e^{x} dx$ (more generally, $\int x^{m} e^{ax} dx$, $m \geq 1$ & integer)
- $\int \ln x \, dx$ (more generally, $\int x^{p} (\ln x)^{m} dx$, $m \geq 1$ & integer, $p =$ any number)

b Theory:

The Product Rule: $(f(x)g(x))' = f \cdot g' + f' g$ implies:

$$f \cdot g' = (f \cdot g)' - f' g .$$

Integrate both sides: $\int f g' dx = \underbrace{\int (f g)' dx} - \int f' g \, dx$

Using $\int F'(x) dx = F(x)$ for ↗ (where we ignore "$+C$"), we get:

$$\underbrace{\int f \cdot g' dx}_{\text{the integral we want to compute}} = f \cdot g - \underbrace{\int f' \cdot g \, dx}_{\text{hoping that this integral is easier that the original one}}$$

Notes:
- This method works sometimes, but not always.
- It may require trial and error.

c Alternative form:

Recall that for any $f(x)$: $\quad df = (f(x))' dx$

Similarly: $\quad dg = (g(x))' dx$

Then: $\int f \underbrace{g'\,dx}_{dg} = f\cdot g - \int g \cdot \underbrace{f'\,dx}_{df}$

$\Rightarrow \quad \int f\,dg = f\cdot g - \int g \cdot df$

The convention is to rename: $f = u$, $g = v$.
Then the previous formula becomes:

$$\boxed{\int u\,dv = u\cdot v - \int v\cdot du}$$

② Integrals $\int x^m \cdot e^{ax}\,dx$, $m \geq 1$ & integer

Ex. 1 = Ex. 6.3.1 in textbook

Find $\int x e^x\,dx$.

Sol'n: Discussion: We can do $\int e^x dx$. So, if we get rid of the "extra" power of x in $\int x\cdot e^x dx$, then we will be able to integrate.

How do we get rid of this x? Note that $x' = 1$. Then:

$$\int \underbrace{x}_{u}\, \underbrace{e^x dx}_{dv} = u\cdot v - \int v\cdot du$$

$u = x \Rightarrow du = \underbrace{x'}_{1}\cdot dx = dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$= x\cdot e^x - \int e^x\cdot dx$$

$$= \boxed{xe^x - e^x + C}$$

Note 1: When choosing dv, as a minimum requirement, we should be able to compute v. (But our goal is to have $\int v\,du$ simpler than $\int u\,dv$.)

Note 2: When computing $v = \int dv$, do not worry about the "$+C$".

Note 3: At the very end, we wrote "$+C$" instead of "$-C$", as the "$-$" in "$-\int e^x dx$" suggest, because C is arbitrary.

Note 4: What happens if we choose "wrong" u & dv?

$$\int x e^x dx = \int \underbrace{e^x}_{u} \cdot \underbrace{x dx}_{dV} = u \cdot v - \int v\, du$$

$u = e^x \Rightarrow du = (e^x)' dx = e^x dx$

$dv = x dx \Rightarrow v = \int x dx = \frac{1}{2} x^2$

$$= e^x \cdot \frac{x^2}{2} - \underbrace{\int \frac{x^2}{2} \cdot e^x dx}$$

this is "worse" than the original $\int x e^x dx$ because the power of x increased, not decreased.

So, again, as noted in topic ①$\boxed{B}$: one should expect trial and error.

Ex. 2 Find $\int (2x+3) e^x dx$.

Sol'n: While this can be done by splitting the integral as: $2\int x e^x dx + 3\int e^x dx$, we will do it "by part" in order to practice this technique.

$$\int \underbrace{(2x+3)}_{u} \underbrace{e^x dx}_{dv} = u \cdot V - \int v \cdot du$$

$u = 2x+3 \Rightarrow du = (2x+3)' dx = 2dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$= (2x+3) \cdot e^x - \int e^x \cdot 2dx$$

$$= (2x+3)e^x - 2e^x + C.$$

Ex. 3 Find $\int x e^{-2x} dx$.

Sol'n: $$\int \underbrace{x}_{u} \underbrace{e^{-2x} dx}_{dv} = u \cdot v - \int v \cdot du$$

$u = x \Rightarrow du = x' dx = dx$

$dv = e^{-2x} dx \Rightarrow v = \int e^{-2x} dx$

Using formula $= -\frac{1}{2} e^{-2x}$

$$\boxed{\int e^{ax} dx = \frac{1}{a} e^{ax}} \;\; \cancel{+C}$$

See p. 3-12 / Lec. 3 (ignore)

$$= x \cdot \left(-\frac{1}{2} e^{-2x}\right) - \int \left(-\frac{1}{2} e^{-2x}\right) \cdot dx$$

$$= -\frac{x}{2} e^{-2x} + \frac{1}{2} \underbrace{\int e^{-2x} \cdot dx}$$

Use the same formula again

$$= -\frac{x}{2} e^{-2x} + \frac{1}{2} \cdot \left(-\frac{1}{2}\right) e^{-2x} + C$$

Ex. 4 Repeated integration by parts ($\approx$ Ex. 6.3.3 in book)

Find $\int x^2 e^x\,dx$.

Sol'n: 1) $\int \underbrace{x^2}_{u} \cdot \underbrace{e^x dx}_{dv} = u\cdot v - \int v\cdot du$

$u = x^2 \Rightarrow du = (x^2)' dx = 2x\,dx$

$dv = e^x \Rightarrow v = \int e^x dx = e^x$

$= x^2 e^x - \int e^x \cdot 2x\,dx$

$= x^2 e^x - 2\underbrace{\int x e^x dx}$

We have lowered the power of x, so we're going in the right direction, simplifying the original integral. In fact, we found the last integral in Ex. 1. Substituting the result from there:

$= x^2 e^x - 2(xe^x - e^x) + C$

$= x^2 e^x - 2xe^x + 2e^x + C$

③ Integrals $\int x^p (\ln x)^m\,dx$, $m \geq 1$ & integer, p = any number

Ex 5 ($\approx$ Ex. 6.3.4 in textbook)

Find $\int \ln x\,dx$.

Sol'n: Discussion: While we don't know how to integrate $\ln x$, we know that $(\ln x)' = \frac{1}{x}$, which is a power function (= something "simpler" than $\ln x$).

So we need to "end up" with $(\ln x)'$ to get rid of $\ln x$.

$\int \underbrace{\ln x}_{u} \cdot \underbrace{dx}_{dv} = u\cdot v - \int v\cdot du = (\ln x)\cdot x - \int \cancel{x}\cdot \frac{1}{\cancel{x}}\,dx$

$u = \ln x \Rightarrow du = (\ln x)' dx = \frac{1}{x}\,dx$

$dv = dx \Rightarrow v = \int dx = x$

$= x\cdot \ln x - \int 1\,dx$

$= x\ln x - x + C$

Ex. 6 Find $\int \ln(2x)\,dx$

Note: $\boxed{\ln(2x) \neq (\ln 2)\cdot x}$

Sol'n: 1) Do a "w-substitution" first. (We previously called it a "u-sub", but here we use "u" for a different purpose.)

$w = 2x \Rightarrow dw = (2x)'\,dx = 2\,dx \Rightarrow dx = \frac{1}{2}\,dw$

Then $\int \ln(2x)\,dx = \int \ln w \cdot \frac{1}{2}\,dw = \frac{1}{2}\int \ln w\,dw$

2) We did the last integral in Ex. 5 (where we had variable x instead of w). So: $\int \ln(2x)\,dx = \frac{1}{2}\int \ln w\,dw =$

$= \frac{1}{2}(w\ln w - w) + C = \frac{1}{\cancel{2}}\left(\cancel{2}x\ln(2x) - \cancel{2}x\right) + C = x\ln(2x) - x + C$

Ex. 7 Find $\int \sqrt{x}\cdot \ln(5x)\,dx$ (Similar to Ex. 6.3.2/book)

Sol'n: Discussion: We have always dealt, in topic ②, with integrals $\int x^{(m = \text{integer})}\cdot e^x\,dx$, since for a non-integer m, integration by parts for those integrals won't work. You can try doing that on your own to see what will happen. (No other methods would work for those integrals, for that matter.) But for $\int x^p \ln x\,dx$, non-integer values of p will work.

Sol'n: 1) As in Ex. 6, we do a "w-sub" first:

$w = 5x \Rightarrow dw = 5dx \Rightarrow dx = \frac{1}{5}dw$ (and also $x = \frac{1}{5}w$).

$$\int \sqrt{x}\,\ln(5x)\,dx = \int \sqrt{\tfrac{1}{5}w}\cdot \ln w \cdot \tfrac{1}{5}dw = \int \sqrt{\tfrac{1}{5}}\cdot\sqrt{w}\cdot \ln w \cdot \tfrac{1}{5}dw$$

$$= \left(\sqrt{\tfrac{1}{5}}\cdot\tfrac{1}{5}\right)\int \sqrt{w}\,\ln w\,dw \underset{\text{Appendix 5}}{=} \left(\tfrac{1}{5}\right)^{3/2}\cdot \int \sqrt{w}\,\ln w\cdot dw$$

So we focus on $\int \sqrt{w}\ln w\,dw$ and will multiply the answer by $\left(\frac{1}{5}\right)^{3/2}$ at the end.

2) $\int \sqrt{w}\cdot \ln w \cdot dw = \int \underbrace{\ln w}_{u}\cdot \underbrace{\sqrt{w}\,dw}_{dv}$

We want $(\ln w)$ "by itself" since we want $\frac{d}{dw}(\ln w) = \frac{1}{w}$.

$u = \ln w \Rightarrow du = (\ln w)'dw = \frac{1}{w}dw$

$dv = \sqrt{w}\,dw \Rightarrow v = \int \sqrt{w}\,dw = \int w^{1/2}\cdot dw = \frac{w^{3/2}}{3/2} = \frac{2}{3}w^{3/2}$

$w^{\frac{3}{2}-1} = w^{\frac{1}{2}}$

So $\int \underset{\ln w}{u}\, \underset{d(\frac{2}{3}w^{3/2})}{dv} = u\cdot v - \int v\,du = \ln w\cdot \frac{2}{3}w^{3/2} - \int \frac{2}{3}w^{3/2}\cdot \frac{1}{w}dw$

$$= \frac{2}{3}w^{3/2}\cdot \ln w - \frac{2}{3}\int w^{1/2}dw = \frac{2}{3}w^{3/2}\cdot \ln w - \frac{2}{3}\cdot \frac{w^{3/2}}{3/2} + C$$

So:

$$\int \sqrt{x}\cdot \ln(5x)\,dx = \left(\tfrac{1}{5}\right)^{3/2}\cdot\left[\tfrac{2}{3}(5x)^{3/2}\cdot \ln(5x) - \left(\tfrac{2}{3}\right)^2\cdot (5x)^{3/2}\right] + C$$

(do not need to simplify)

Ex. 8 Find $\int (\ln x)^2 dx$.

Note: $(\ln x)^2 \neq \ln x^2 \; (= \ln(x^2))$.

Sol'n: $\int \underbrace{(\ln x)^2}_{u} \underbrace{dx}_{dv}$

$u = (\ln x)^2 \Rightarrow du = ((\ln x)^2)' dx$.

$dv = dx \Rightarrow v = x$

Do $((\ln x)^2)'$ by **Chain Rule** (Sec. 3.4, Thm. 2):

If $f(x) = \underset{\uparrow \atop \text{external}}{E}(\underset{\nwarrow \atop \text{internal}}{I}(x))$,

then:

$$\frac{df}{dx} = \frac{dE}{dI} \cdot \frac{dI}{dx}$$

$f(x) = (\underbrace{\ln x}_{I(x)})^2 \quad E(I) = I^2$

$$\frac{d}{dx}(\ln x)^2 = \frac{dE}{dI} \cdot \frac{dI}{dx} = \frac{d\,I^2}{dI} \cdot \frac{d \ln x}{dx} = 2I \cdot \frac{1}{x} = 2(\ln x) \cdot \frac{1}{x}.$$

Then

$du = 2(\ln x) \cdot \frac{1}{x} \cdot dx$. So:

$$\int \underbrace{(\ln x)^2}_{u} dv = u \cdot v - \int v \cdot du = (\ln x)^2 \cdot x - \int \cancel{x} \cdot 2(\ln x) \cdot \frac{1}{\cancel{x}} dx$$

$$= x \cdot (\ln x)^2 - 2\int \ln x \cdot dx$$

Not only did we lower the power of $\ln x$, but we also have previously found this $\int$ (Ex. 5).

$$\overset{\text{Ex. 5}}{=} x(\ln x)^2 - 2(x \ln x - x) + C$$

Ex. 9 A cool integral

Find $\int \frac{\ln x}{x}\,dx$.

Sol'n: Method 1 = By Parts: $= \int \underbrace{\ln x}_{u} \cdot \underbrace{\frac{1}{x}\,dx}_{dv}$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = \frac{1}{x}\,dx \Rightarrow v = \int \frac{dx}{x} = \ln x$

$$\underbrace{\int \ln x \cdot \frac{1}{x}\,dx}_{\uparrow} = \underbrace{\ln x}_{u} \cdot \underbrace{\ln x}_{v} - \underbrace{\int \underbrace{\ln x}_{v} \cdot \underbrace{\frac{1}{x}\,dx}_{du}}_{\uparrow} \quad \ldots$$

Same integral...
So, no progress?

Wait. Call $\int \ln x \cdot \frac{1}{x}\,dx = I$ ← the integral we want to find.

Then we showed above: $I = (\ln x)^2 - I \Rightarrow$

$I + I = (\ln x)^2 \Rightarrow 2I = (\ln x)^2 \Rightarrow I = \frac{1}{2}(\ln x)^2 + \underbrace{C}_{\text{as usual}}$.

Method 2: u-sub.

$\int \ln x \cdot \frac{1}{x}\,dx$ Let $u = \ln x \Rightarrow$
$du = \frac{1}{x}\,dx$

$= \int u \cdot du$

$= \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

Same answer as Method 1, but easier <u>in this case</u>