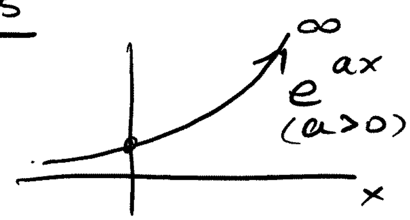
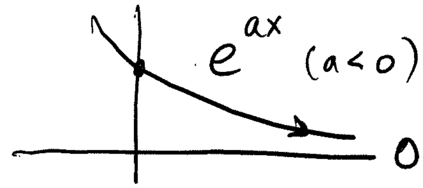
Lecture 6 – Improper integrals

① Reminder of important limits

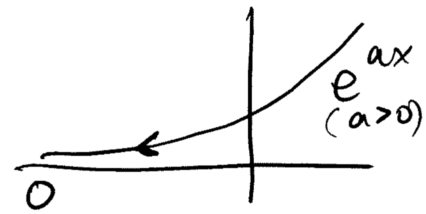
L1.a $\lim\limits_{x\to\infty} e^{ax} = \infty$ for $a>0$



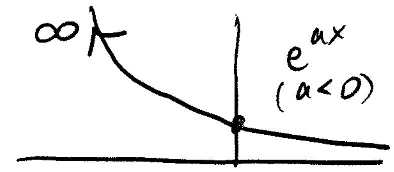
L1.b $\lim\limits_{x\to\infty} e^{ax} = 0$ for $a<0$



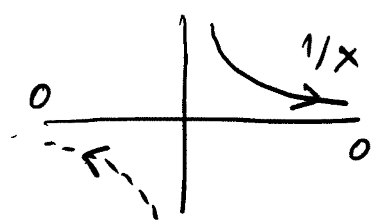
L1.c $\lim\limits_{x\to-\infty} e^{ax} = 0$ for $a>0$



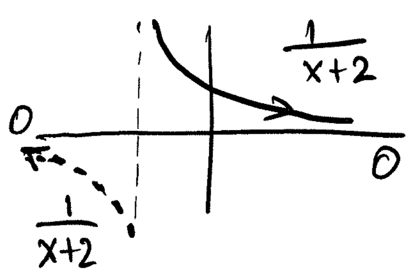
L1.d $\lim\limits_{x\to-\infty} e^{ax} = \infty$ for $a<0$



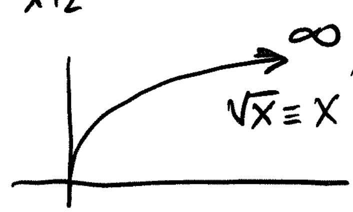
L2.a $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ for $n>0$



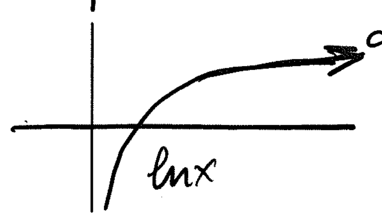
L2.b $\lim\limits_{x\to\infty} \frac{1}{(x+a)^n} = 0$ for $n>0$ and <u>any</u> $a=\text{const}$



L3 $\lim\limits_{x\to\infty} x^n = \infty$ for $n>0$



L4 $\lim\limits_{x\to\infty} \ln x = \infty$



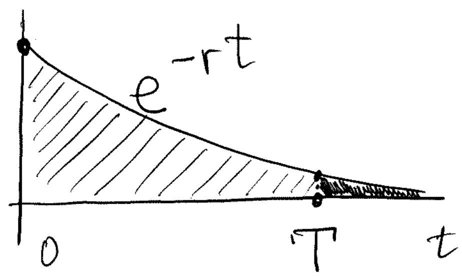
② Motivation & definition of improper integrals

Ex. 1 In Lec. 3 when we discussed Future Value (FV) (see topic ①, p. 4), we had an integral of the form

$$\int_0^T e^{-rt}\,dt = \frac{1-e^{-rT}}{r}$$

Discuss what happens when T is very large ($T \to \infty$).

Discussion: 1) $\displaystyle\int_0^T e^{-rt}\,dt = \begin{pmatrix}\text{area under } e^{-rt} \\ \text{over } [0, T]\end{pmatrix}$



We can think intuitively that as T is getting larger, the area to the right of T (shaded darker) gets smaller and likely tends to 0. This means that as we "move" T to the right, the light-shaded area changes less and less and eventually tends to a constant limit.

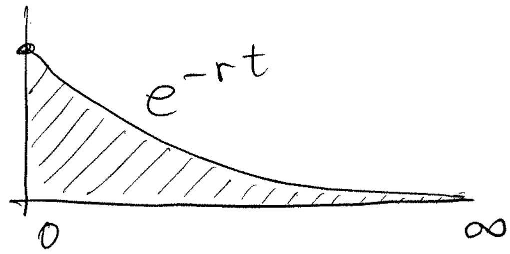
Let us verify this guess by a calculation.

2) $$\lim_{T\to\infty} \int_0^T e^{-rt}\,dt \underset{\substack{\uparrow \\ \text{use the above} \\ \text{formula}}}{=} \lim_{T\to\infty} \frac{1-e^{-rT}}{r}$$

$$= \frac{1}{r}\lim_{T\to\infty}(1-e^{-rT}) = \frac{1}{r}\Big(1-\underbrace{\lim_{T\to\infty} e^{-rT}}_{\to\, 0 \text{ by } \boxed{L1.b}}\Big)$$

$$= \frac{1}{r}(1-0) = \frac{1}{r}$$

3) Thus, we can say:



$$\begin{pmatrix} \text{area under } e^{-rt} \\ \text{over } [0, \infty) \end{pmatrix} = \lim_{T\to\infty} \int_0^T e^{-rt}\,dt = \frac{1}{r}$$

Defining $\displaystyle\int_0^\infty e^{-rt}\,dt \overset{\text{def.}}{=} \lim_{T\to\infty} \int_0^T e^{-rt}\,dt$,

we see that $\boxed{\displaystyle\int_0^\infty e^{-rt}\,dt = \frac{1}{r}}$

This motivates the definitions:

Def. 1 $\displaystyle\int_a^\infty f(t)\,dt = \lim_{b\to\infty} \int_a^b f(t)\,dt$

Def. 2 $\displaystyle\int_{-\infty}^b f(t)\,dt = \lim_{a\to-\infty} \int_a^b f(t)\,dt$

Def. 3 $\displaystyle\int_{-\infty}^\infty f(t)\,dt = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(t)\,dt$

Provided that any of the limits above exists, the corresponding integral with "∞" and/or "$-\infty$" as a bound is called the <u>improper integral</u>.

When the improper integral <u>exists</u>, it is said to <u>converge</u>; otherwise it is said to <u>diverge</u>.

Ex. 2 Find $\int_{-\infty}^{0} e^{-2\cdot x}\,dx$

Sol'n: 1) $\int_{-\infty}^{0} e^{-2x}\,dx = \lim_{c\to-\infty} \int_{c}^{0} e^{(-2)x}\,dx =$

using formula $\boxed{\int e^{ax}dx = \frac{1}{a}e^{ax}}$ $a=-2$

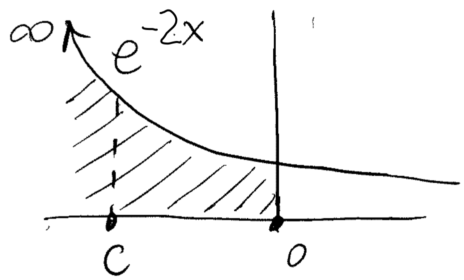
$= \lim_{c\to-\infty} \frac{1}{-2}\, e^{x}\Big|_{c}^{0}$

$= -\frac{1}{2}\lim_{c\to-\infty}\left(e^{(-2)\cdot 0} - e^{(-2)\cdot c}\right)$

$= -\left(1 - \lim_{c\to-\infty} e^{(-2)c}\right) \overset{\boxed{L\,1.d}}{=} -(1-\infty) = \infty.$

So, this integral diverges (does not exist).

2) Why? Let us look at the graph:



As $c\to-\infty$, the area under the e^{-2x} increases to ∞ because $e^{-2x} \not\to 0$.

Based on this example we formulate **a criterion**:

$$\boxed{\text{If } \lim_{x\to\infty} f(x) \neq 0 \;\Rightarrow\; \int_{a}^{\infty} f(x)\,dx \text{ diverges}}$$

same

(And similarly for $\int_{-\infty}^{b}$ and $\int_{-\infty}^{\infty}$).

Important: $\lim_{x\to\infty} f(x) = 0 \;\not\Rightarrow\; \int_{a}^{\infty} f(x)\,dx$ converges

(see topic ④ below).

3 Some applications of improper integrals

a Ex. 3 Production of a commodity ($\approx$ Ex. 11.1.5)

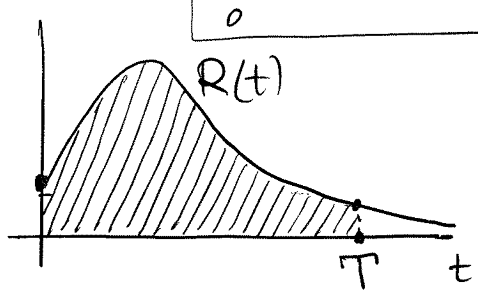
It is estimated that an oil well will produce oil at a rate $R(t) = 9e^{-0.05t} - 8e^{-0.1t}$

"Rate"

(in million barrels per year), where t is the time since the opening of the well. Estimate the total amount of oil to be produced by this well.

Sol'n: 1) = Discussion.

The amount of oil produced by the well over time T is: $\boxed{\int_0^T R(t)\,dt}$.



As you know from Lec. 3, running an operation incurs maintenance cost (topic 3 in Lec. 3), and so the well will be closed soon after it stops generating profit. So, the time T to which we need to integrate will be finite, not infinite. However, as a rough estimate, we can evaluate the above integral for $T \to \infty$ (since it is slightly easier than for a finite T). This will give us an over-estimate of the total amount of oil produced, but it will still be in the right ballpark.

So, our estimate will be:

$$\boxed{\begin{pmatrix}\text{Total amount}\\ \text{of oil produced}\end{pmatrix} \approx \int_0^\infty R(t)\,dt}$$

2) $\int_0^\infty R(t)\,dt = \int_0^\infty (9e^{-0.05t} - 8e^{-0.1t})\,dt$

$= 9\int_0^\infty e^{-0.05t}\,dt - 8\int_0^\infty e^{-0.1t}\,dt \overset{\downarrow}{=}$ formula on p. 6-3

$= \frac{9}{0.05} - \frac{8}{0.1} = 100$ (million barrels).

$\boxed{b}$ Capital value of the income stream

Theory.

- From Lec. 3, topic 1, p. 3-7, we know that if income is deposited continuously at a rate $f(t)$ and if the interest rate on the deposit is r, then the future value of this income stream at time T is:

$$FV = \int_0^T f(t)e^{r(T-t)}\,dt = \int_0^T f(t)\underbrace{e^{rT}}_{\text{const}}\cdot e^{-rt}\,dt = e^{rT}\cdot\int_0^T f(t)e^{-rt}\,dt$$

- On the other hand, one can ask: What one-time deposit, P_0, can one make at $t=0$ so that it results at the same future value FV (if the interest rate is the same, r)?

We know that for a one-time deposit P_0:

$$FV = P_0 e^{rT}$$

- Equating the two expressions, one gets:

$$\cancel{e^{rT}}\int_0^T f(t)e^{-rt}\,dt = P_0\cancel{e^{rT}}$$

i.e., $P_0 = \int_0^T f(t)e^{-rt}\,dt.$

- We can now pretend that the income stream exists for a long time and we are interested in a distant future, i.e. $T \to \infty$. Then we define:

$$\boxed{CV = \int_0^\infty f(t) e^{-rt}\, dt}$$

Capital Value of income stream $f(t)$ that brings compounded interest at a rate r.

Ex. 4 (= Ex. 11.1, 6)

The owner of an oil well has leased it to a larger oil company in return for a perpetual (= continuing indefinitely) annual payment of \$120,000. Find the Capital Value of this lease at the interest rate 5% compounded continuously.

Sol'n: 1) We are given that: $r = 5\%$, $f(t) = 120,000$ (\$/year).

2) $CV = \int_0^\infty 120,000 \cdot e^{-0.05t}\, dt =$

$$= 120,000 \cdot \int_0^\infty e^{-0.05t}\, dt = \frac{120,000}{0.05} = \$2,400,000$$

formula from p. 6-3

4) Improper integrals of x^n (and the like)

First, we note that if $n > 0$, then $\lim_{x \to \infty} x^n = \infty$

(see $\boxed{L3}$ on p. 6-1). Then, since $\lim\limits_{x\to\infty} x^{(n>0)} \neq 0$,

then by the criterion at the end of p. 6-4,

$\int_a^\infty x^n\,dx$ **diverges** (does not exist) for $n>0$.

Ex. 5 For what n does $\int_1^\infty \frac{dx}{x^n}$ converge?

Sol'n: $\int_1^\infty \frac{dx}{x^n} = \lim\limits_{b\to\infty}\int_1^b \frac{dx}{x^n} = \lim\limits_{b\to\infty}\int_1^b x^{-n}\,dx$

a) $n>1$

$$= \lim_{b\to\infty} \left.\frac{x^{-n+1}}{-n+1}\right|_1^b = \lim_{b\to\infty} \frac{b^{-n+1}}{-n+1} - \underbrace{\frac{1}{-n+1}}_{\text{const}}$$

$$\underset{\substack{\uparrow\\ \text{Appendix 5}}}{=} \lim_{b\to\infty} \underbrace{\frac{1}{(-n+1)}\cdot\frac{1}{b^{(n-1)>0}}}_{\to 0} + \left(\frac{1}{n-1}\right) = 0 + \frac{1}{n-1}$$

since $n>1$

$$= \frac{1}{n-1}.$$

Thus, for $n>1$, $\int_1^\infty \frac{dx}{x^n}$ (and $\int_{a>0}^\infty \frac{dx}{x^n}$) **converges** (exists).

b) $n<1$ The same calculation yields

$$\int_1^\infty \frac{dx}{x^n} = \lim_{b\to\infty}\left(\frac{1}{-n+1}\right)\cdot\frac{1}{b^{(n-1)<0}} + \underbrace{\frac{1}{n-1}}_{\text{const}}$$

since $n<1$

$$= \underbrace{\lim_{b\to\infty}\left(\frac{1}{-n+1}\right) b^{(1-n)>0}}_{\boxed{L3},\ \text{p. 6-1}} + \text{const} = \infty + \text{const} = \infty$$

Thus, for $\underline{\underline{n < 1}}$, $\int_1^\infty \frac{dx}{x^n}$ $\left(\text{and } \int_{a>0}^\infty \frac{dx}{x^n}\right)$ **diverges** (does not exist).

c) $\underline{n = 1}$

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big]_1^b = \lim_{b\to\infty} (\ln b - \cancel{\ln 1}^{\,0})$$

$$= \lim_{b\to\infty} \ln b = \infty \quad \boxed{L4}, \text{ p. 6-1.}$$

Thus, for $\underline{\underline{n=1}}$, $\int_1^\infty \frac{dx}{x^n} \equiv \int_1^\infty \frac{dx}{x}$ $\left(\text{and } \int_{a>0}^\infty \frac{dx}{x}\right)$ **diverges**.

IN SUMMARY:

$$\boxed{\int_a^\infty \frac{dx}{x^n} \; (a>0) \quad \begin{cases} \text{converges for } n > 1 \\ \text{diverges for } n \le 1 \end{cases}}$$

Note 1 By a u-sub: $u = x + c$, one can show that $\int_a^\infty \frac{dx}{(x+c)^n}$ for $c > 0$ satisfies the same properties.

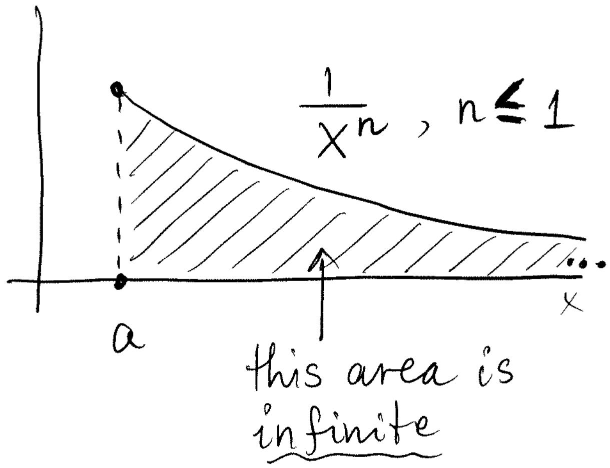
Note 2 On p. 6-4 (bottom) we said that:

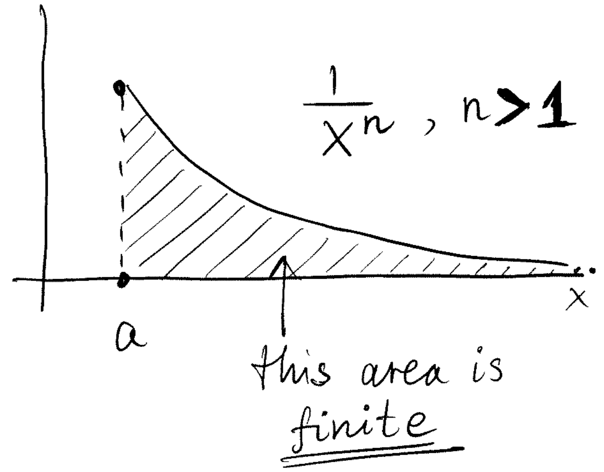
$$\lim_{x\to\infty} f(x) = 0 \quad \cancel{\Rightarrow} \quad \left(\int_1^\infty f(x)\,dx \text{ converges}\right)$$

Indeed, $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, but $\int_1^\infty \frac{dx}{x}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ both diverge.

On the other hand, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, and $\int_1^\infty \frac{dx}{x^2}$ converges.

Note 3 We can interpret the above result as:





Ex. 6 Find $\int_2^\infty \frac{x\,dx}{(3+x^2)^2}$

Sol'n: 1) Use a u-sub to reduce this to Ex. 5.

$u = 3 + x^2$

$du = (3+x^2)'\,dx = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$u(2) = 3 + 2^2 = 7$

$u(\infty) = 3 + \infty^2 = \infty$

$$\overset{u(\infty)}{=} \int_{\underset{u(2)}{7}}^{\infty} \frac{\frac{1}{2}\,du}{u^2} = \frac{1}{2}\int_7^\infty \frac{du}{u^2}$$

$n = 2 \Rightarrow$ by Ex. 5 this converges.

2) Do without "$\frac{1}{2}$" and multiply by it at the end.

$$\int_7^\infty \frac{du}{u^2} = \lim_{b\to\infty} \int_7^b u^{-2}\,du = \lim_{b\to\infty} \left.\frac{u^{-1}}{-1}\right|_7^b =$$

$$= \lim_{b\to\infty}\left(\frac{b^{-1}}{-1} - \frac{7^{-1}}{-1}\right) = \lim_{b\to\infty}\left(-\frac{1}{b}\right) + \frac{1}{7} = 0 + \frac{1}{7}.$$

3) $$\int_2^\infty \frac{x\,dx}{(3+x^2)^2} = \frac{1}{2}\int_7^\infty \frac{du}{u^2} = \frac{1}{2}\cdot\frac{1}{7} = \frac{1}{14}$$ //