# Lecture 9 — Exponential and Normal PDFs.

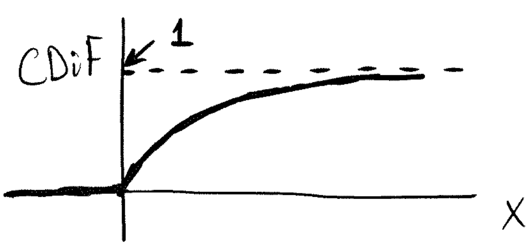
## 1 Exponentially distributed random variables

We have already encountered exponential PDFs, so below is a summary, with some new emphases:

- An exponential PDF is: $f(t) = \begin{cases} \frac{1}{\lambda} e^{-t/\lambda}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

  ("$\lambda$" is the Greek letter "lambda"; it is the only parameter of the exponential PDF, and its significance will be established soon.)

- CDiF is: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 1 - e^{-x/\lambda}, & x \geq 0 \\ 0, & x < 0 \end{cases}$


Derivation (for $x \geq 0$)

$$F(x) = \int_0^x \frac{1}{\lambda} e^{-t/\lambda}\,dt \qquad \begin{aligned} u &= -\frac{t}{\lambda} \\ du &= -\frac{dt}{\lambda} \\ dt &= -\lambda\,du \\ u(x) &= -\frac{x}{\lambda} \\ u(0) &= 0 \end{aligned}$$

$$= \int_{u=0}^{u=-x/\lambda} \frac{1}{\cancel{\lambda}} e^u \cdot (-\cancel{\lambda}\,du)$$

$$= -\int_0^{-x/\lambda} e^u\,du =$$

$$= -e^u \Big|_0^{-x/\lambda} = -(e^{-x/\lambda} - e^0) = 1 - e^{-x/\lambda}.$$

- Mean: $\mu = \int_0^\infty t \cdot \frac{1}{\lambda} e^{-t/\lambda}\,dt = \lambda$ (Skip work)

- Variance $= \int_0^\infty t^2 \cdot \frac{1}{\lambda} e^{-t/\lambda}\,dt - \mu^2 = \underset{\text{calculations}}{\text{lots of}} \ldots = \lambda^2$

- Standard deviation: $\sigma = \sqrt{\text{Variance}} = \lambda$
- Median: $F(m) = \frac{1}{2} \Rightarrow 1 - e^{-m/\lambda} = \frac{1}{2} \Rightarrow 1 - \frac{1}{2} = e^{-m/\lambda}$

$$\Rightarrow \frac{1}{2} = e^{-m/\lambda} \Rightarrow \ln\frac{1}{2} = -\frac{m}{\lambda} \Rightarrow \cancel{-}\ln 2 = -\frac{m}{\lambda} \Rightarrow$$

$$m = \lambda \cdot \ln 2$$

<u>Ex. 1</u> (= Ex. 11.4.3 in text book)

Time between phone calls is an exponential random variable. The average time between calls is 20 seconds. What is the probability that two consecutive calls come in more than 30 seconds apart?

<u>Sol'n:</u> 0) Exponential PDFs are characterized by a single parameter, $\lambda$. So we need to first extract $\lambda$ from the information given.

1) $\mu = 20$ sec, $\Rightarrow \lambda = \mu = 20$ (sec.)

2) $P(X \geq 30) = ?$

We know that $P(X \leq 30) \equiv F(30) = 1 - e^{-x/\lambda}\Big|_{\substack{x=30 \\ \lambda=20}}$

$= 1 - e^{-30/20} = 1 - e^{-3/2}$.

How are $\underbrace{P(X \leq 30)}_{\text{known}}$ and $\underbrace{P(X > 30)}_{\text{wanted}}$ related?

$$\boxed{P(X \leq 30) + P(X > 30) = 1} \quad (*)$$

Therefore $P(X > 30) = 1 - P(X \leq 30) = \cancel{1} - (\cancel{1} - e^{-3/2})$

$= e^{-3/2} \approx 0.22$

**Ex. 2** The life expectancy of a part of a computer is an exponential random variable. 40% of these parts fail in the first 3 years. The computer manufacturer offers a 1-year warranty. What is the probability that this part fails within the warranty period?

Sol'n: 0) Again, the exponential random variable is characterized by a single parameter $\lambda$, so the first step is to extract $\lambda$ from the given information.

1) $P(X \le \underbrace{3}_{\text{years}}) = 0.4 \; (=40\%)$

$F(3) = 0.4 \Rightarrow 1 - e^{-3/\lambda} = 0.4 \Rightarrow \overbrace{1-0.4}^{0.6} = e^{-3/\lambda}$

$\ln 0.6 = -3/\lambda \Rightarrow \lambda = -3/\ln 0.6 \approx 5.87 \text{ (years)}$

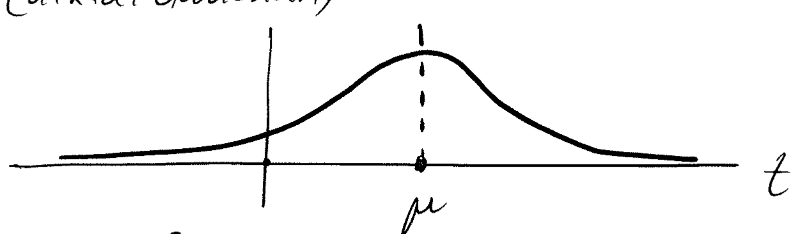
2) Use this $\lambda$ to answer the question of the problem:

$P(X \le \underbrace{1}_{1\text{ year}}) = F(1) = 1 - e^{-1/\lambda} = 1 - e^{-1/5.87} \approx 0.16$

$= 16\%$

## 2 NORMALLY distributed random variables

Shape of the normal (a.k.a. Gaussian) distribution is:




Formula:

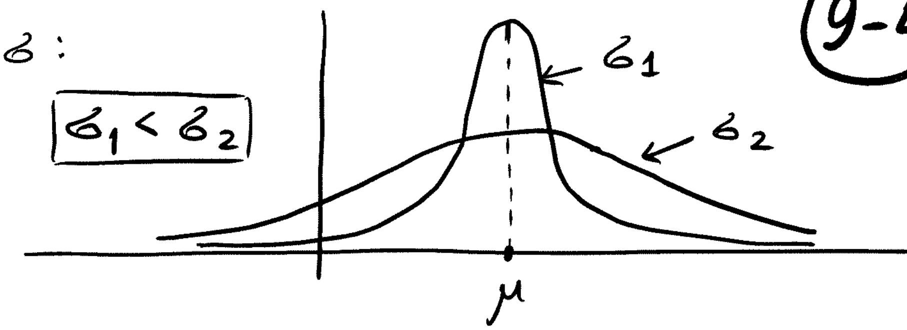
$$f(t) = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

st. dev. $\to \sigma$

Shape varies with $\sigma$:

**Larger $\sigma$ means a more spread out PDF.**

$\boxed{\sigma_1 < \sigma_2}$




**Facts:**

- Normal distributions are very common.
- $\int_{-\infty}^{\infty} f(t)\,dt = 1$ (Shown by methods beyond this course.)
- $\mathbf{E(X)} = \int_{-\infty}^{\infty} t \cdot f(t)\,dt = \boldsymbol{\mu}$
- Variance $= \int_{-\infty}^{\infty} (t-\mu)^2 f(t)\,dt = \sigma^2 \Rightarrow$ **St. dev. = $\sigma$.**
- CDiF: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$

Cannot be expressed as "known" functions.
Tabulated (in Table 2 in Appendix C of textbook) using the following form:

$$\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt = \int_{-\infty}^{z} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \cdot \cancel{\sigma}\,du$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du}_{\textbf{This function depends only on } z\textbf{!}}$$

Tabulated in Appendix C.

$u = \frac{t-\mu}{\sigma}$

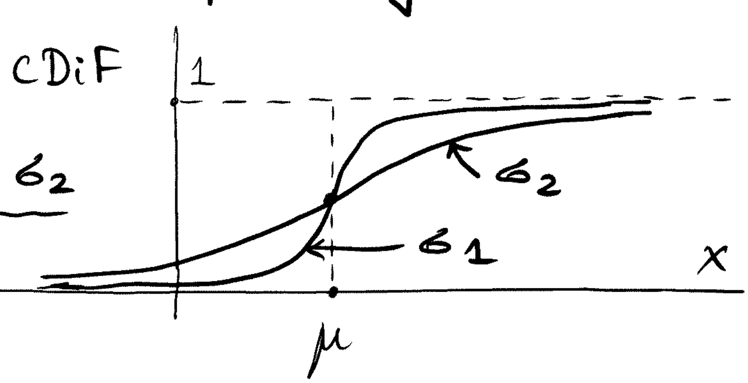
$du = \frac{dt - 0}{\sigma}$

$dt = \sigma \cdot du$

$u(-\infty) = -\infty$

$u(x) = \frac{x-\mu}{\sigma} \equiv z$

$\sigma_1 < \sigma_2$



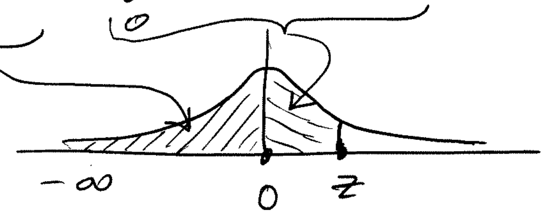
- $$\mathcal{P}(c \le X \le d) \overset{\text{Lec. 7, p. 7-12}}{=} F(d) - F(c)$$

$$= \int_{-\infty}^{d} f(t)\,dt - \int_{-\infty}^{c} f(t)\,dt = \int_{-\infty}^{z_d} \tilde{f}(u)\,du - \int_{-\infty}^{z_c} \tilde{f}(u)\,du,$$

where: $\tilde{f}(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$, $z_d = \frac{d-\mu}{\sigma}$, $z_c = \frac{c-\mu}{\sigma}$.

For convenience of using this formula with Table 2 of Appendix C, we'll simplify it now.

Writing $$\int_{-\infty}^{z} \tilde{f}(u)\,du \equiv \underbrace{\int_{-\infty}^{0} \tilde{f}(u)\,du} + \underbrace{\int_{0}^{z} \tilde{f}(u)\,du}$$



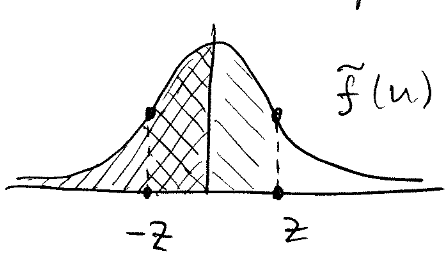
rewrite the previous formula as:

$$\mathcal{P}(c \le X \le d) = \left(\cancel{\int_{-\infty}^{0} \tilde{f}(u)\,du} + \int_{0}^{z_d} \tilde{f}(u)\,du\right) - \left(\cancel{\int_{-\infty}^{0} \tilde{f}(u)\,du} + \int_{0}^{z_c} \tilde{f}(u)\,du\right)$$

$$= \int_{0}^{z_d} \tilde{f}(u)\,du - \int_{0}^{z_c} \tilde{f}(u)\,du \equiv G(z_d) - G(z_c)$$ for Gauss

**Here $G(z) = \int_0^z \tilde{f}(u)\,du$ is the function tabulated in Appendix C.**

- Above we defined $G(z)$ for $z > 0$. What is $G(\underbrace{-z}_{<0})$?



$$\int_{-\infty}^{-z} \tilde{f}(u)\,du = \int_{-\infty}^{0} \tilde{f}(u)\,du - \int_{0}^{z} \tilde{f}(u)\,du$$

!!!

See the picture and note that $\tilde{f}(u)$ is symmetric about $u=0$.

Therefore: $$G(-z) = -G(z).$$

- What is the probability to be within $(n \cdot \sigma)$ from $\mu$?
(We'll consider $n = 1, 2, 3$.)

$$P(\mu - n\sigma \le X \le \mu + n\sigma) = ?$$

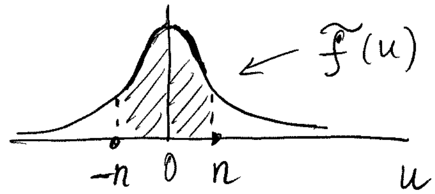
1) Compute $z_d = \frac{(\cancel{\mu} + n\cdot\sigma) - \cancel{\mu}}{\sigma} = \frac{n\cdot\cancel{\sigma}}{\cancel{\sigma}} = n \Rightarrow \mathbf{z_d = n};$

$$z_c = \frac{(\cancel{\mu} - n\cdot\sigma) - \cancel{\mu}}{\sigma} = \frac{-n\cdot\cancel{\sigma}}{\cancel{\sigma}} = -n = -z_d$$

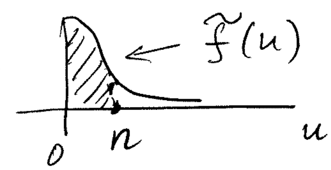
2) Apply the formulas from the previous page:

$$P(\mu - n\sigma \le X \le \mu + n\sigma) = G(z_d) - G(z_c) =$$
$$= G(z_d) - G(-z_d) = G(z_d) - (-G(z_d)) = 2\cdot G(z_d)$$

Graphical illustration:




$= 2 \cdot$




Using the numbers from Table 2/Appendix C for $z = 1, 2, 3$:

**Probability to be within $n\cdot\sigma$ of $\mu$, i.e. in $(\mu - n\sigma, \mu + n\sigma)$:**

$n = 1 \rightarrow P(\mu - 1\cdot\sigma \le X \le \mu + 1\cdot\sigma) \approx 68\%$

$n = 2 \rightarrow P(\mu - 2\cdot\sigma \le X \le \mu + 2\cdot\sigma) \approx 95\%$

$n = 3 \rightarrow P(\mu - 3\cdot\sigma \le X \le \mu + 3\cdot\sigma) \approx 99.7\%$

__Ex. 3__ (= Ex. 11.4.4 & 5 in textbook)

A factory produces lightbulbs with life expectancies that are normally distributed with a mean of 500 hr and a standard deviation of 100 hr. What percentage of the lightbulbs can be expected to last between 380 and 670 hrs?

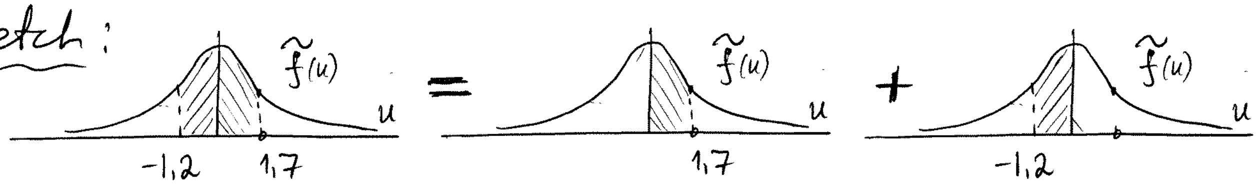
Sol'n: 1) Start with the general formula (p. 9-5):

$$P(c \le X \le d) = G(z_d) - G(z_c)$$

$$z_d = \frac{d-\mu}{\sigma} = \frac{670-500}{100} = 1.7, \quad z_c = \frac{c-\mu}{\sigma} = \frac{380-500}{\sigma} = -1.2$$

$$P(c \le X \le d) = G(1.7) - G(-1.2) = G(1.7) - (-G(1.2)) = G(1.7) + G(1.2).$$

Sketch:

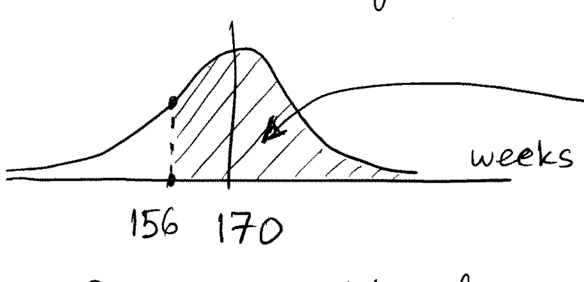



2) Use the Table from Appendix C:

$G(1.7) \approx 0.46$, $G(1.2) \approx 0.38$ $\Rightarrow$ $P(380 \le X \le 670) = 84\,\%$.

**Ex. 4** The life expectancy of a car battery is a normally distributed random variable. The average life time is 170 weeks with a standard deviation of 10 weeks. If the company guarantees the battery for 3 years, what percentage of the batteries is expected to last past their warranty?

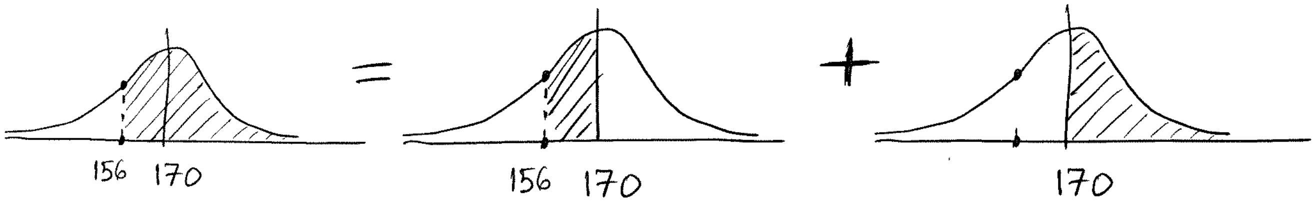
Sol'n: 1) This problem is different from that in Ex. 3, and so it is a good idea (as always!) to start with a sketch.




Warranty = 3 years = $3 \cdot 52 = 156$ weeks

These are the batteries in question.

2) Based on the figure, we have:



3) For the first term, we have $c = 156$, $d = 170$

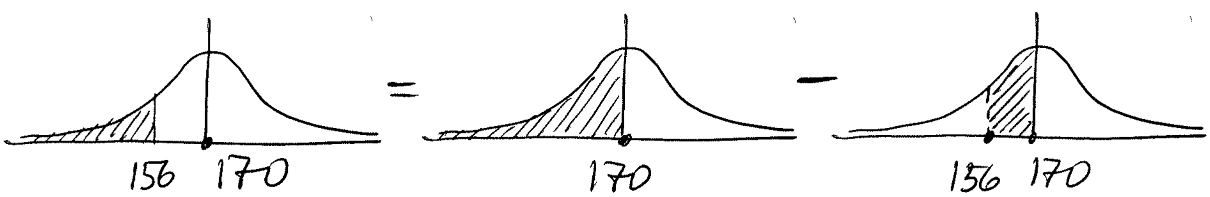
$$z_d = \frac{170-170}{10} = 0, \quad z_c = \frac{156-170}{10} = -1.4$$

$$\underline{P}(156 \le \underline{X} \le 170) = G(0) - G(-1.4) = 0 - (-G(1.4))$$
$$= G(1.4) \approx 0.42$$

4) For the second term: We note that this is exactly half the total area, so it is $\frac{1}{2} \cdot 1 = 1/2$.

5) Thus, $\underline{P}(X \geq 156) \approx 0.42 + 0.50 = 0.92 = 92\%$.

Note: If the problem had asked: What percentage of the batteries fail within the warranty period, then one could proceed starting with a picture:




$$= 1/2 - (G(0) - G(-1.4)) = 1/2 + G(-1.4) = 1/2 - G(1.4)$$
$$= 0.50 - 0.42 = 0.08 = 8\%.$$

Expectedly enough, this is the same as $100\% - 92\%$.