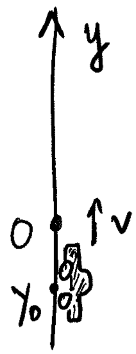
Lecture 10 — Differential equations.
Introduction, and what to expect of solutions.

① Motivation



- Consider a car moving along the y-axis (maybe, in the North–South direction ☺) with velocity v (= const).
  If at $t=0$ the car is at $y=y_0$, where will it be at a later time t?

Answer: $y = y_0 + v \cdot t$

Let us take the derivative (w.r.t. time):

$$y' = v \tag{1}$$

and record the initial location: $y(0) = y_0$. (2)

Equation (1), or

$$\frac{dy}{dt} = v \tag{1*}$$

is a simple example of a Differential Equation.
Equation (2) is called the initial condition.

We can tell if the car moves South → North or North → South based on whether $v > 0$ or $v < 0$:

$v > 0$: $y' = v \Rightarrow y' > 0 \Rightarrow y \nearrow$ (South → North)

$v < 0$: $y' = v \Rightarrow y' < 0 \Rightarrow y \searrow$ (North → South)

And if $v = 0$, the car's position remains constant:

$v = 0$: $y' = v \Rightarrow y' = 0 \Rightarrow y$ **does not change**.

- Now what if the velocity of the car changes in time:

$v \equiv v(t) \leftarrow$ some known function of t ?

One still has Eq. (1*):

$$\frac{dy}{dt} = v(t) \qquad (1^*)$$

because, by definition, the (instantaneous) velocity is the rate of change of the position.
To solve for $y(t)$, we **take the anti-derivative** (= indefinite **integral**) of **both sides**:

$$y(t) = \int v(t)\,dt . \qquad (3)$$

<u>Ex. 1(a)</u> Suppose that on some time interval, the velocity of a car is: $v(t) = 30 + 2t$ (in some units). Find the position $y(t)$ of this car.

<u>Sol'n:</u> By Eq. (3):

$$y = \int (30 + 2t)\,dt = 30t + t^2 + C$$

We cannot determine the <u>exact</u> position because we do not know C!

<u>Ex. 1(b)</u> Solve the previous problem if it is further known that $y(0) = 10$.

<u>Sol'n:</u> $y(t) = 30t + t^2 + C$.

Using $y(0) = 10 \Rightarrow 30 \cdot 0 + 0^2 + C = 10 \Rightarrow C = 10$.

So: $y(t) = 30t + t^2 + 10$.

② What to expect in this Chapter

- We will be solving differential equations of the form:

$$\frac{dy}{dt} = f(t, y), \qquad (4)$$

where $f(t, y)$ is some known function.

Notes:
- y is called the dependent variable
  t is called the independent variable

To solve a diff. eq. means to find how the solution y depends on t.

- In the textbook, they often use "x" instead of "t".
- We will not be able to use the simple formula given by Eq. (3) on p. 10-2 to solve Eq. (4). Indeed, if we write:

$$\frac{dy}{dt} = f(t,y) \Rightarrow y = \int f(t,y)\,dt \equiv \int f(t,y(t))\,dt \qquad (5)$$

we see that

we cannot integrate (5) because we do not know $y(t)$!

Example: In applications one often deals with $f(t,y) = a\cdot y + b$, where a, b are some constants (so, this $f(t,y)$ does not actually depend on t explicitly; it does so only implicity via $y(t)$).
So, if we write:

$$\frac{dy}{dt} = a\cdot y + b \Rightarrow y = \int (ay+b)\,dt = a\int y(t)\,dt + b\cdot t,$$

we see that we cannot find $\int y(t)\,dt$ because we do not know $y(t)$!

So, **we will need to learn some methods to solve** Eq. (4).
For now, remember: **Never "solve" $\frac{dy}{dt} = f(t,y)$ by**

$$\cancel{\frac{dy}{dt} = \int f(t,y)\,dt} \leftarrow$$ **correct formula, but WRONG METHOD!**

- Equation (4) arises in many (thousands) applications. E.g., $y(t)$ can represent the population of some organisms; then

$$\frac{dy}{dt} = f(t, y)$$

is interpreted as:

$$\begin{pmatrix}\text{rate of change}\\ \text{of the population}\end{pmatrix} = \begin{pmatrix}\text{some known function of}\\ \text{time and the current population}\end{pmatrix}.$$

As another example, $y(t)$ can be the amount of an investment, or loan, etc...

- Even though $\frac{dy}{dt} = \int f(t,y)\,dt$ is a wrong method to solve our diff. eq., it is still a correct formula. It is useless for finding $y(t)$ (see p. 10-3), but it shows that **the solution** is expressed as an indefinite integral and therefore **always contains an arbitrary constant C.**

  Caveat: This "C" does not always enter the solution as "+C". See Ex. 2 below.

In other words:

The solution of a diff. eq. is not unique; it is determined up to that arbitrary constant C. (Will see many examples.)

- To find "that C", we used the initial condition $y(0) = y_0$. Therefore:

The solution of a diff. eq. along with the initial condition is unique! (See Ex. 1(b).)

- Although we will learn some methods of solving $dy/dt = f(t,y)$, keep in mind that this equation does not have an "analytical solution" (i.e., a solution given by a formula) for "most" functions $f(t,y)$.
So, what we will learn here will be some important, but special cases of how to solve the above Eq. (4).
In most practical cases, one solves the combination "diff. eq. PLUS the initial condition" on a computer and obtains the solution as a graph $y = y(t)$.

- Even if we cannot solve $\frac{dy}{dt} = f(t,y)$ to obtain a formula for its solution, we can approximately sketch $y(t)$ based on the following:
  - If $f(t,y) > 0 \Rightarrow \frac{dy}{dt} > 0 \Rightarrow y(t) \nearrow$
  - If $f(t,y) < 0 \Rightarrow \frac{dy}{dt} < 0 \Rightarrow y(t) \searrow$.

③ Verifying a solution of the diff. eq. and finding the arbitrary constant C.

Ex. 2(a) Verify that $y = Cx^2 + 1$ is the general solution of the diff. eq. $xy' = 2y - 2$. (Here $y' = dy/dx$.)

Note 1: The above y is called the "general" solution of the diff. eq. because by varying the constant C, one can get all possible solutions of that diff. eq.

Note 2: The purpose of this Example is to demonstrate that "the constant C" can enter the solution differently than as "$+C$", as in Ex. 1(a).

Sol'n:

To verify any equation, we:

1) Substitute it into the left-hand side (l.h.s.) of the eq.;
2) Substitute it into the right-hand side (r.h.s.) of the eq.;
3) Compare the results of 1) & 2), which should match.

1) l.h.s. $= xy' = x\cdot(Cx^2+1)' = x\cdot(C\cdot 2x+0) = C\cdot 2x^2$.

2) r.h.s. $= 2y-2 = 2(Cx^2+1)-2 = 2Cx^2+\cancel{2}-\cancel{2} = 2Cx^2$.

3) lhs = rhs ✓

Ex. 2(b) Using the general solution from Ex. 2(a), find a particular solution of the diff. eq.

$xy' = 2y-2$ (same as in Ex. 2(a))

that satisfies the given initial conditions:

(i) $y(1) = 3$.

Substitute $y = Cx^2+1$ with $x=1$ to the l.h.s.:

$C\cdot 1^2+1 = 3 \Rightarrow C = 2$.

So $y = 2\cdot x^2+1$.

(ii) $y(0) = 3$

Similarly, $C\cdot 0^2+1 = 3 \Rightarrow 0+1 = 3$... No solutions!

(iii) $y(0) = 1$

Similarly, $C\cdot 0^2+1 = 1 \Rightarrow 1 = 1 \Rightarrow$ any C will work

$\Rightarrow$ $y = Cx^2+1$ with any C satisfies this initial cond.

<u>Discussion of the strange results in Ex. 2(B)(ii, iii)</u>

Equation $xy' = 2y - 2$ is <u>not</u> of the form

$$y' = f(x,y).$$

To make it so, we must isolate y', so divide by x:

$$\frac{\cancel{x}y'}{\cancel{x}} = \frac{2y-2}{x} \Rightarrow y' = \underbrace{\frac{2y-2}{x}}_{f(x,y)}$$

We see that $f(x=0, y)$ is undefined.

And, it was precisely for the initial condition set at $x=0$ where we saw the strange behaviors: either no solutions or infinitely many solutions.

In the rest of this Chapter we will always work with $f(x,y)$ (or $f(t,y)$) which are defined for all x (or t) and all y, and therefore **<u>all</u> our remaining differential equations will <u>always</u> have a <u>unique particular solution</u> when an initial condition has been specified.**

(4) Setting up differential equations for some applications

<u>Ex. 3</u> <u>Simple population growth model</u>

Let P be the population (i.e., number of individuals) of organisms (people, animals, bacteria) whose evolution obeys these simple rules:

- The population can change only due to either births or deaths within it (i.e., no emigration or immigration).

— The rates at which birth and death occur are proportional to the current value of the population.

Derive the differential equation for P.

Sol'n:

1) $\begin{pmatrix}\text{Rate of change}\\ \text{of } P\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{due to births}\end{pmatrix} - \begin{pmatrix}\text{Rate of change}\\ \text{due to deaths}\end{pmatrix}$

2) It is given that:

(Rate of change due to births) $= r_b \cdot P$.

r_b is the proportionality constant called the birth rate.

Similarly:

(Rate of change due to **deaths**) $= r_d \cdot P$.

r_d is the proportionality constant called the death rate.

3) Combining 1) & 2):

$$\underbrace{\frac{dP}{dt}}_{\text{Rate of change of } P} = r_b \cdot P - r_d \cdot P$$

We can factor the common P on the r.h.s.:

$$\frac{dP}{dt} = \underbrace{(r_b - r_d)}_{\text{call this "}a\text{"}} \cdot P \Rightarrow \boxed{\frac{dP}{dt} = a \cdot P}$$

This equation is called the **exponential growth model** (because it has exponentially growing or decaying solutions, as we will show). It is probably the most important differential equation that you need to know.

Ex. 4 Continuously compounded interest with income flow rate

Money is continuously deposited to an account at a rate of \$1000/year. The account earns 7% annual interest rate, compounded continuously. Derive the differential equation for the amount of money on the account. (Denote this amount by A.)

Sol'n:

1) $\begin{pmatrix}\text{Rate of change}\\ \text{of } A\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{due to income flow}\end{pmatrix} + \begin{pmatrix}\text{Rate of change}\\ \text{due to interest}\\ \text{compounding}\end{pmatrix}$

2) Rate of change due to income flow = \$1000/year.
Rate of change due to interest compounding $= r \cdot A$, where r is the annual percent rate (Sec. 1.5).

3) Combining 1) & 2):

$$\underbrace{\frac{dA}{dt}}_{\text{Rate of change of } A} = 1000 + \overbrace{r}^{0.07 \text{ in this example}} \cdot A.$$

Ex. 5 Mortgage repayment

You take a loan in a bank to buy a house. Every month you pay a constant amount, **m**, of mortgage. The bank charges you interest (continuously) on the amount of loan remaining; this annual interest rate is **r**. Derive the differential equation for the loan **L** that remains to be paid to the bank.

Sol'n: 1)

$$\begin{pmatrix}\text{Rate of change of} \\ \text{unpaid loan}\end{pmatrix} = -\begin{pmatrix}\text{mortgage} \\ \text{payments}\end{pmatrix} + \begin{pmatrix}\text{Rate of accumulation} \\ \text{of interest on} \\ \text{unpaid loan}\end{pmatrix}$$

2) mortgage payments $= m$ (\$/year)

Rate of interest accumulation on unpaid loan $= r \cdot L$

3) Combining 1) & 2):

$$\underbrace{\frac{dL}{dt}}_{\text{Rate of change of unpaid loan}} = -m + r \cdot L$$

(5) Sketching the solution: increase, decrease, equilibrium, behavior at infinite time

Here we will use the observation from p. 10-5:

If $dy/dt = f(t,y)$ and

- $f(t,y) > 0 \Rightarrow y' > 0 \Rightarrow y \nearrow$;
- $f(t,y) < 0 \Rightarrow y' < 0 \Rightarrow y \searrow$.

Ex. 6 (a) Exponential growth model of Ex. 3

Verify that $P = Ce^{at}$ is the general solution of the population model in Ex. 3. Discuss the behavior of the solution depending on the sign of a. In particular, what happens to the solution when $t \to \infty$?

Sol'n: 1) Following Ex. 2(a), compare the l.h.s. with the r.h.s. in $\frac{dP}{dt} = a \cdot P$.

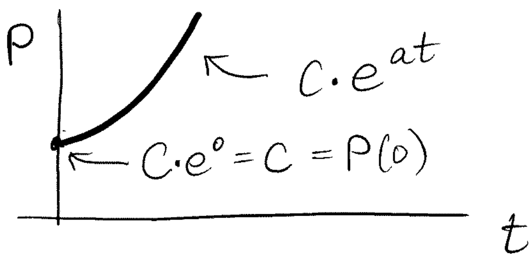
$$\underline{lhs} = \frac{d\,(C \cdot e^{at})}{dt} = C \cdot (e^{at})' = C \cdot a \cdot e^{at}$$

$\underline{rhs} = a \cdot P = a \cdot C \cdot e^{at}$. Indeed, $\underline{rhs} = \underline{lhs}$. ✓

2) Graphs of e^{at} are found in Sec. 1.5 and Lecture 6, p. 6-1.

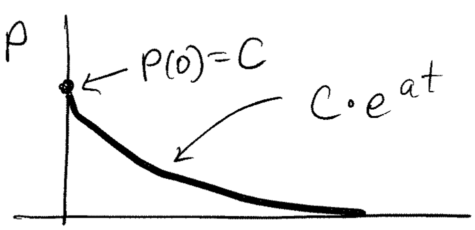
MUST KNOW!!

$\underline{a>0}$ (since $a = r_b - r_d \Rightarrow r_b > r_d$)



So, when $a>0$, or births dominate deaths, the solution grows (exponentially). As $t \to \infty$, $P \to \infty$.

$\underline{a<0}$ ($r_b < r_d$)



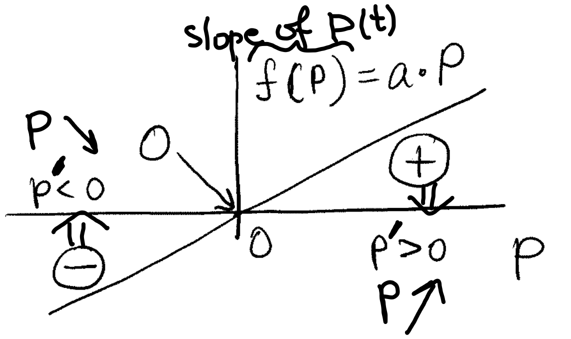
When deaths dominate births, the solution decays to zero (exponentially). As $t \to \infty$, $P \to 0$.

Ex. 6(b) Re-obtain these results using the observation stated at the beginning of this topic (about ↗ or ↘).

Sol'n:

Case $a>0$ $\qquad P' = aP = \overbrace{f(P)}^{\text{slope of } P(t)}$ (doesn't depend on t)

1) Plot $f(P)$ vs. P and identify regions where $f(P)<0$, $f(P)=0$, $f(P)>0$:



By the reminder at the beginning of this topic:

$f(P) > 0 \Rightarrow P' > 0 \Rightarrow P \nearrow$

$f(P) < 0 \Rightarrow P' < 0 \Rightarrow P \searrow$

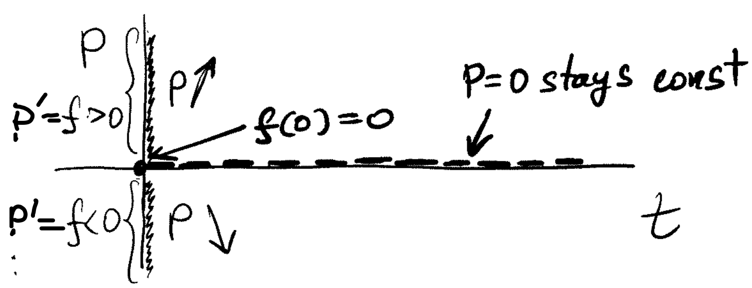
$f(P) = 0 \Rightarrow P = \text{const.}$

2) — Make a new plot with axes t and P:

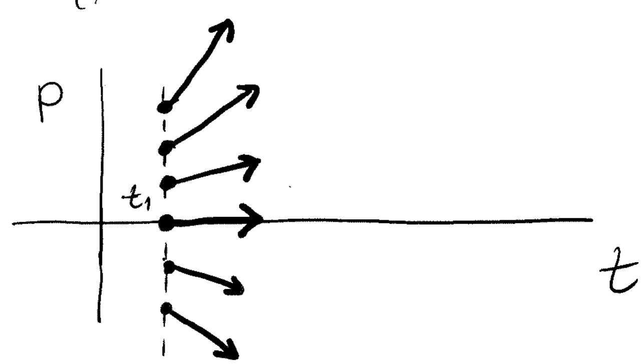
Note that the axes of this plot are different from the one above!

In the previous plot we plotted $f(P)$ vs. P. Here, we will plot (or, rather, sketch) P vs. t.

— In this new plot, label **along the P-axis** the regions where $f(P) < 0$, $f(P) = 0$, $f(P) = 0$:



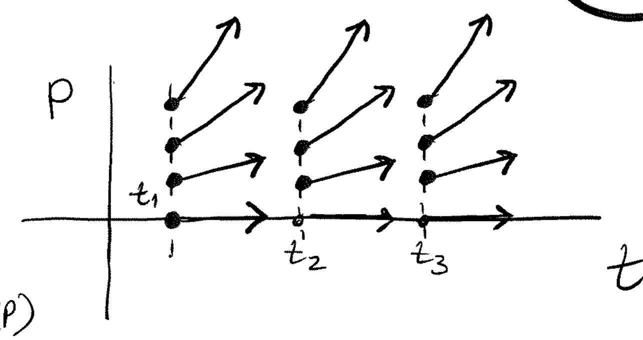
— In each part, where $P \nearrow$ or $P \searrow$, pick a value of t (any) and sketch arrows whose slopes



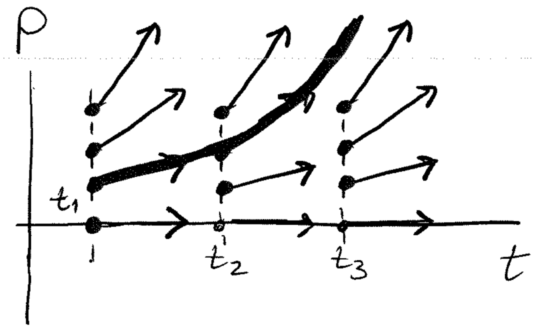
qualitatively agree with $f(P) = P'$ at some two or three values of P. That is, the higher $f(P)$, the higher the slope should be.

(Since P must be > 0, ignore $P < 0$ for this problem.)

– Repeat plotting exactly the same arrows for a couple of other values of t:
(same, because their slope $f(P)$ does not depend on t).

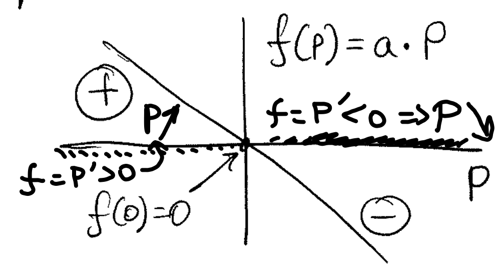


– Connect some of these arrows by some smooth trajectory, making sure that **the trajectory is tangent to the arrows.**
Such a trajectory depicts one of the solutions.

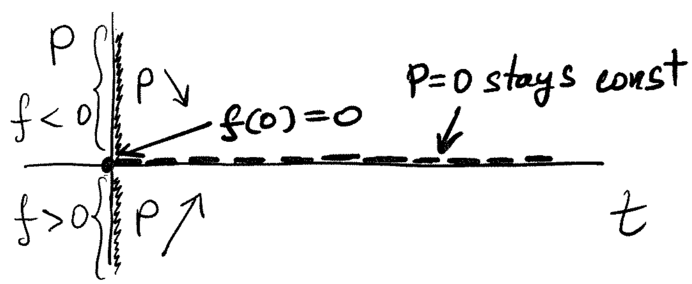


Case $a<0$ Repeat the same steps:

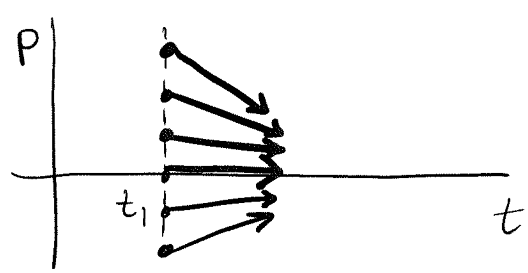
– Sketch $f(P)$ vs P, paying attention to the sign.



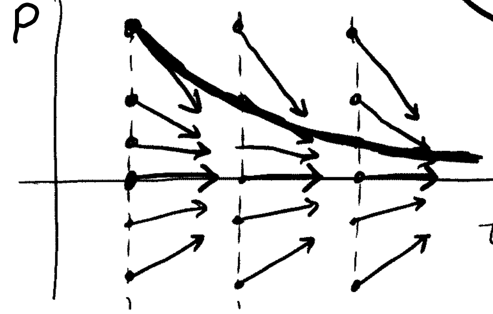
– Make a new plot with axes P vs t and **label**, **along the P axis**, the regions where $P \nearrow$ or $\searrow$.



– At any one value of t, draw arrows whose slope (positive/negative, large/small) agrees with the value of $f(P)$.



— Repeat this for a couple of values of t and then connect some of the arrows by a trajectory, making sure that the trajectory is tangent to the arrows.



Note that the so-sketched trajectories in Ex. 6(b) agree with the graphs of the analytical sol'n, $y = Ce^{at}$, in Ex. 6(a).

Ex. 7 <u>Repeating Ex. 6 for the Mortgage Repayment Model of Ex. 5.</u>

(a) Verify that $L = \frac{m}{r} + Ce^{rt}$ is the (general) solution of diff. eq.:

$$\frac{dL}{dt} = -m + r \cdot L. \quad \left(\begin{array}{l} L = \text{unpaid loan}, \\ m = \text{mortgate payment} \\ r = \text{interest rate} \end{array}\right)$$

Establish its behavior at $t \to \infty$.

<u>Sol'n:</u>

$$\underline{lhs} = \frac{dL}{dt} = \frac{d}{dt}\Big(\overset{\text{const}}{\frac{m}{r}} + Ce^{rt}\Big) = 0 + C \cdot re^{rt}$$

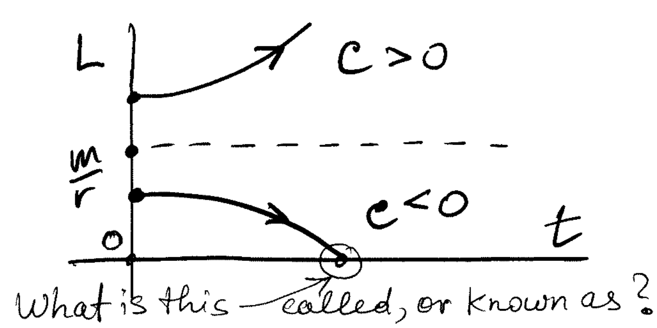
$$\underline{rhs} = -m + r \cdot \Big(\frac{m}{r} + Ce^{rt}\Big) = -m + \cancel{r} \cdot \frac{m}{\cancel{r}} + r \cdot Ce^{rt}$$

$$= \cancel{-m} + \cancel{m} + r \cdot Ce^{rt} = r \cdot Ce^{rt} = \underline{lhs}. \checkmark$$

Since $r > 0$, $e^{rt}\Big|_{t \to \infty} \to \infty$,

and $C \cdot e^{rt} \to +\infty$ for $C > 0$

and $\to -\infty$ for $C < 0$.

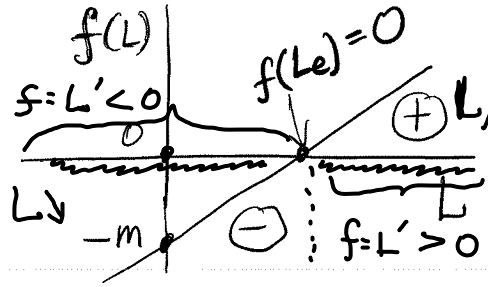


(b) Reobtain the sketches found in part (a) by using the information about where $L \nearrow$ or $\searrow$.

<u>Sol'n:</u> Follow the steps of Ex. 6(b).

— For $\frac{dL}{dt} = \underbrace{-m + rL}_{f(L)}$,

sketch $f(L)$ vs L, paying attention to the signs.



Note a new notation L_e; "e" stands for "equilibrium".

When $L = L_e$, $f(L_e) = 0$, $\Rightarrow \left.\frac{dL}{dt}\right|_{L=L_e} = f(L_e) = 0$.

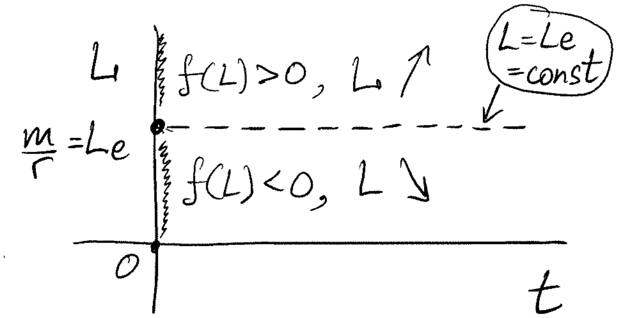
So this equilibrium solution <u>remains constant in time</u>.

The <u>equilibrium solution</u> is a <u>very important</u> concept.

We can find L_e here: $f(L_e) = 0 \Rightarrow -m + r \cdot L_e = 0$

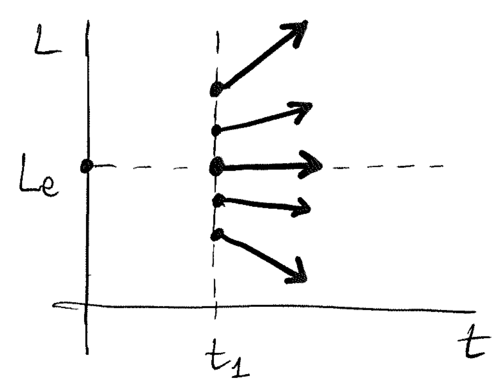
$\Rightarrow \boxed{L_e = m/r}$.

— Plot axes L vs t and label, along the L axis, regions where $f(L) > 0, = 0$, and < 0.

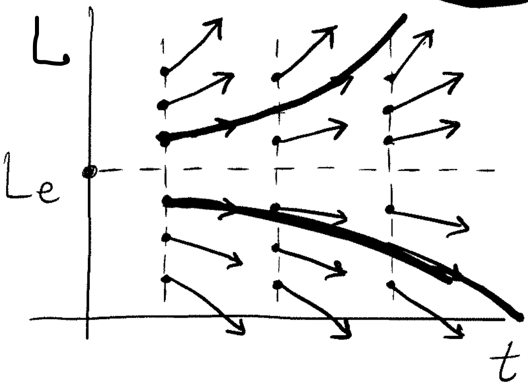


(In this problem, $L < 0$ does not make sense, so we focus on the positive part of the L-axis.)

— For any t_1, plot several arrows at different L-values. The slope of the arrows must represent the size & sign of $f(L)$.



— Repeat this for a couple of other t-values and connect the arrows by a trajectory.
This trajectory must be tangent to the arrows.
In the figure on the right we plot one trajectory <u>above</u> $L = L_e$ and another trajector <u>below</u> $L = L_e$.



Note that if $L = L_e$ initially (or at any time), then L remains L_e for all times.

> Therefore, a trajectory (= the graph of a solution of the diff. eq.) <u>can</u> <u>NEVER</u> <u>cross</u> $L = L_e$.