Lecture 11 — Solving separable DEs

⓪. Reminder

Before you attempt to solve any DE, put it in the form $y' = f(x,y)$ (or $y' = f(t,y)$).

Ex. 1(a) Write $y' + 2y = 3x + 4$ in the above form.

$$y' = \underbrace{3x+4-2y}_{f(x,y) \text{ for this DE.}}$$

Ex. 1(b) Same for $y' + 2y = 4$.

$$y' = \underbrace{4-2y}_{f(x,y)} \leftarrow \text{This } f \text{ does not explicitly depend on } x,$$

but it is OK.

① Separable DEs and their solution

1a The DE of the form

$$y' = \frac{g(x) \leftarrow \text{depends only on } x}{h(y) \leftarrow \text{depends only on } y}$$

for some $g(x)$ and $h(y)$ is called **separable**.

It is solved as follows.

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow h(y)\,dy = g(x)\,dx \Rightarrow$$

$$\boxed{\int h(y)\,dy = \int g(x)\,dx} \leftarrow \text{This is the (implicit) solution of the original DE.}$$

$\boxed{1B}$ What of $y' = g(x)\cdot w(y)$
(for some $g(x)$ and $w(y)$)?

Proceed similarly:

$$\frac{dy}{dx} = g(x)\cdot w(y) \Rightarrow \frac{dy}{w(y)} = g(x)\,dx \Rightarrow$$

$$\int \frac{dy}{w(y)} = \int g(x)\,dx.$$

② Solution of the simple exponential growth/decay model

In topic ④ of Lec. 10 we encountered the model

$$\frac{dP}{dt} = a\cdot P, \quad a = \text{const},$$

which describes evolution of populations (P) due to births and deaths. This same model arises in countless other applications, a glimpse of which we will see in this Lecture and accompanying HWs.

Let us re-denote $P \equiv y$. Then:

$$\frac{dy}{dt} = a\cdot y \Rightarrow \frac{dy}{y} = a\cdot dt \Rightarrow \int \frac{dy}{y} = \int a\cdot dt$$

$$\Rightarrow \ln|y| = at + \underbrace{C_1}_{\text{arb. const}} \Rightarrow e^{\ln|y|} = e^{at + C_1}$$

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$$\Rightarrow \underbrace{|y|}_{>0} = \underbrace{e^{at}}_{>0}\cdot \underbrace{e^{C_1}}_{\equiv C > 0} \Rightarrow |y| = C\cdot e^{at}.$$

To get rid of the $|\dots|$, we consider $y>0$ & $y<0$ separately.

$\underline{y>0} \Rightarrow y = C \cdot e^{at}$ (with $C>0$)

$\underline{y<0} \Rightarrow -y = C \cdot e^{at} \Rightarrow$

$y = (-C) \cdot e^{at}$ (with $(-C)<0$)

Both of these cases can be expressed by one formula:

$$\boxed{y = C \cdot e^{at}} \leftarrow \text{general solution of } y' = a \cdot y$$

where now C can be >0 or <0 (depending on the initial condition).

③ Solution of the limited growth/decay model

The name of this model will be justified when we consider its applications. The model is:

$$\frac{dy}{dt} = ay + b, \quad a, b = \text{const.}$$

$$y' = a\left(y + \frac{b}{a}\right) \Rightarrow \frac{dy}{y + \left(\frac{b}{a}\right)} = a \cdot dt \Rightarrow \int \frac{dy}{y + \frac{b}{a}} = \int a\,dt$$

$\Rightarrow \ln\left|y + \frac{b}{a}\right| = at + C_1 \Rightarrow$ Proceeding exactly as in topic ②:

$$y + \frac{b}{a} = C \cdot e^{at} \Rightarrow$$

Solution of the limited growth/decay model $\rightarrow$ $\boxed{y = Ce^{at} - \frac{b}{a}}$, C is a constant determined by the initial condition.

4) Separable DEs with non-exponential soln's

We will provide two examples of such DEs. In fact, there are infinitely many DEs with non-exponential solutions. But when we later consider <u>applications</u>, we will mostly deal with exponential solutions, because they are one of the most important.
So, for this course, non-exponential solutions will be somewhat "exotic".

<u>Ex. 2</u> Find the solution of $y' = y^2$, $y(0) = \frac{1}{3}$.

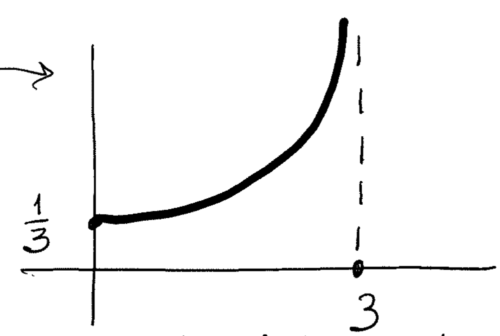
<u>Sol'n:</u> 1) $\frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{y^2} = dx \Rightarrow \int \frac{dy}{y^2} = \int dx$

$\Rightarrow \int y^{-2} dy = \int dx \Rightarrow -\frac{1}{y} = x + C \Rightarrow y = -\frac{1}{x+C}$

2) Find C to match the initial condition:

$y(0) = \frac{1}{3} = -\frac{1}{0+C} \Rightarrow 3 = -C \Rightarrow C = -3.$

$y = -\frac{1}{x-3} = \frac{1}{3-x}$ $\longrightarrow$



Interestingly, unlike the exponential solution, which takes (formally) an infinite amount of time to go to ∞, the solution of this example "explodes" in a finite time (3 units).

Ex. 3 Find the solution of $y' = \frac{2y}{x}$, $y(1) = -3$.

Sol'n:

1) $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} = \frac{2\,dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{2\,dx}{x}$

$$\Rightarrow \ln|y| = 2\ln|x| + C_1 \Rightarrow e^{\ln|y|} = e^{2\ln|x| + C_1}$$

$$\Rightarrow |y| = e^{2\ln|x|} \cdot \underbrace{e^{C_1}}_{\equiv C > 0} = \left(e^{\ln|x|}\right)^2 \cdot C = \left(|x|\right)^2 \cdot C = C \cdot x^2.$$

Appendix A.5

2) Find C from the initial condition $y(1) = -3$:

Since $y|_{initial} < 0$, we use $|y| = -y$:

$$\underbrace{-y}_{>0} = \underbrace{C}_{>0} \cdot \underbrace{x^2}_{>0} \Rightarrow y = \underbrace{-C}_{<0} \cdot x^2 \equiv \underbrace{C}_{} \cdot x^2$$

rename $(-C)$ as C; expect it now to be negative

@ $x = 1$:

$-3 = C \cdot 1^2 \Rightarrow -3 = C$

$C < 0$, as expected!

Answer: $y = -3x^2$.

Note: You absolutely have to know the rule $e^{\ln a} = a$ and the Properties of Exponents listed in Theorem 1 in Appendix A.5.

⑤ Applications of the simple exponential growth/decay model

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Ex. 4 Law of radioactive decay

$\boxed{\text{Radioactive materials decay at a rate proportional to the amount of the material present at a given moment.}}$ ← The Law.

Denoting the amount of material by m, this Law can be stated as the DE:

$$\frac{dm}{dt} = \underbrace{-k}_{<0} \cdot m \qquad \left(\text{Since } m \searrow, \Rightarrow \frac{dm}{dt} < 0. \right)$$

$k\ (>0)$ is called the **decay constant.**

Carbon-14 (^{14}C), an isotope of carbon, has $k = 1.2 \cdot 10^{-4}$ 1/year. At what time will 50 % of the initially present amount of ^{14}C decay?

This time is called the **half-life** of the given material and is usually denoted as T.

Sol'n: 1) The equation $\frac{dm}{dt} = -k \cdot m$ has the form of $y' = a \cdot y$ with $a = -k$ (and $y = m$). State the formula for the general solution (topic ②):

$$m = C \cdot e^{-kt}$$

2) $m(0) = C \cdot e^0 = C$.

At the half-life time, we have:

$$m(T) = \frac{1}{2} m(0).$$

Using the solution from 1), this yields:

$$\cancel{C} \cdot e^{-kT} = \frac{1}{2} m(0) = \frac{1}{2} \cancel{C} \Rightarrow -kT = \ln \frac{1}{2}$$

$$\Rightarrow \boxed{T = \frac{\ln 2}{K} \iff K = \frac{\ln 2}{T}}$$

If one knows the decay constant, one also knows the half-life, and vice versa.

3) Substitute the value of K:

$$T = \frac{\ln 2}{1.2 \cdot 10^{-4}} \approx 5730 \text{ years.}$$

This is the time in which 50% of the initially present ^{14}C decays.

Ex. 5 The amount of an active ingredient in a desinfectant added to a swimming pool diminishes (due to evaporation) at a rate proportional to the amount of this ingredient still remaining in the pool. 30% of this ingredient evaporates in 3 days. The desinfectant is considered ineffective when 90% of the ingredient has evaporated.
How long does the desinfectant remain effective?

Given: $m(3) = (1-0.3) \cdot m(0)$
(m = amount of the ingredient)

Find: t_{end} s.t. $m(t_{end}) = (1-0.9)\, m(0)$.

Sol'n: 1) State the model and its solution:

$$\frac{dm}{dt} = \underbrace{-K}_{<0,\ \Rightarrow K>0} \cdot m. \qquad m \searrow \text{ (given)} \Rightarrow \frac{dm}{dt} < 0.$$

$$m(t) = C \cdot e^{-Kt}$$

2) Use the Given information to find K:

$m(3) = 0.7\, m(0)$

$\cancel{C} e^{-k\cdot 3} = 0.7 \cdot \cancel{C} \cdot \cancel{e^{0}}^{1} \Rightarrow \ln e^{-k\cdot 3} = \ln 0.7$

$-k\cdot 3 = \ln 0.7 \Rightarrow k = \dfrac{-\ln 0.7}{3} \approx 0.119$

3) Answer the question of the problem:

$m(t) = 0.1 \cdot m(0)$

$\cancel{C} \cdot e^{-k\cdot t} = 0.1 \cdot \cancel{C} \Rightarrow -kt = \ln 0.1 \Rightarrow$

$t = -\dfrac{\ln 0.1}{k} \approx 19.4$

So, the desinfectant needs to be re-applied in 19-20 days.

⑥ Applications of the limited growth/decay model

Ex. 6 Newton's Cooling Law

When the gallon of milk is taken out of a 40°F refrigerator, it warms up proportionally to the difference between its current temperature and the room's temperature of 72°F. In 0.5 hr, the milk's temperature is 60°F.

(a) What is it temperature in 1 hr?

(b) How long will it take to warm up to 70°F?

Sol'n:

1) State the model for milk's temperature T:

Step 1: $\dfrac{dT}{dt} = \underbrace{k}_{\text{"proportional to"}} \cdot (T - 72)$

<u>Step 2</u>: Determine the sign of k.

The milk <u>warms up</u> from $T < 72$. So $dT/dt > 0$.

Then must have $k \cdot \underbrace{(T-72)}_{<0 \text{ since } T<72} > 0 \Rightarrow \boxed{k < 0}$

The convention is to re-denote $\underbrace{k}_{<0} \to \underbrace{-k}_{>0}$,

so that the "new k" > 0. Then

$$\frac{dT}{dt} = -k \cdot (T-72), \quad \text{with some } k>0 .$$

To put it in the form used in topic 3 :

$$\frac{dT}{dt} = \underbrace{-k}_{a} \cdot T + \underbrace{k \cdot 72}_{b},$$

$\Rightarrow$ from topic 3, the solution is :

$$T = -\frac{b}{a} + Ce^{-kt} = \cancel{-}\frac{\cancel{k}\cdot 72}{\cancel{-}\cancel{k}} + Ce^{-kt} = \mathbf{72 + Ce^{-kt}} .$$

2) Use the Given information to find C and k:

- $T(0) = 40 = 72 + C \cdot \cancel{e^{0}}^{1} \Rightarrow 40 = 72 + C \Rightarrow C = -32$
- $T(0{,}5) = 60 \Rightarrow 60 = 72 - 32 \cdot e^{-k \cdot 0{,}5}$

$$60 - 72 = -32e^{-0{,}5k} \Rightarrow \frac{\cancel{-}12}{\cancel{-}32} = e^{-0{,}5k} \Rightarrow \ln\frac{12}{32} = -0{,}5k$$

$$k = \frac{\ln(12/32)}{-0{,}5} \approx 1.96$$

3) Finish the problem:

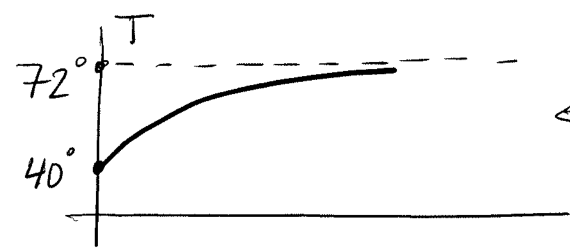
$$T(1) = 72 - 32 \cdot e^{-1{,}96 \cdot 1} = 67{,}5\ ^\circ F$$

(b) $T(t) = 70 = 72 - 32e^{-1{,}96t} \Rightarrow \overbrace{70-72}^{-2} = -32e^{-1{,}96 \cdot t}$

$\Rightarrow \ln \frac{\cancel{-}2}{\cancel{-}32} = -1{,}96t \Rightarrow t = \frac{\ln(2/32)}{-1{,}96} \approx 1{,}4\,hr$

$= 1\,hr\ 25\,min.$

Graph:



← Limited growth

Ex. 7 Paying off a mortgage

Disclaimer: This is not a limited growth/decay model, but uses the same mathematics. We consider this problem due to its obvious real-life significance.

Problem: In Ex. 5 of Lec. 10 we derived the DE for the loan L remaining on a mortgage:

$$\frac{dL}{dt} = \underbrace{-m}_{\text{mortgage payments}} + r \cdot L$$

r: APR on the loan. L: unpaid loan

(a) Find the general solution to this equation.

(b) If the principal on the house (= initial loan amount) is $P_0 = \$400K$, APR $= 6\%$, and the mortgage is for $T = 30$ years, what should your yearly/monthly payments to the bank be?

(c) How much money in total will you pay the bank for your house?

Sol'n: (a)

From topic 3: $\frac{dL}{dt} = \underset{a}{r} \cdot L + \underbrace{(-m)}_{b} \Rightarrow$

$$L = -\frac{b}{a} + Ce^{at} = -\frac{(-m)}{r} + Ce^{rt} = \frac{m}{r} + Ce^{rt}$$

(b) We know:

$\underbrace{L(0) = P_0 = 400K}$; $\underbrace{L(T) = 0}$ (loan paid off in T years)

$$P_0 = \frac{m}{r} + C\cancel{e^{0}}^{\,1}$$

$$C = 400K - \frac{m}{r}$$

$$0 = \frac{m}{r} + C \cdot e^{r \cdot T}$$

$$0 = \frac{m}{r} + \left(400K - \frac{m}{r}\right)e^{rT}$$

$$\boxed{0} = \frac{m}{r}(1 - e^{rT}) + \underbrace{400K}_{P_0} \cdot e^{rT}$$

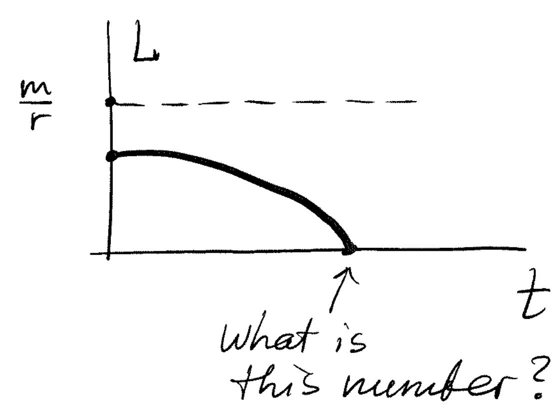
$$\frac{m}{r}(e^{rT} - 1) = P_0 \Rightarrow m = r \cdot \frac{P_0 e^{rT}}{e^{rT} - 1}$$

Substitute given values:

$$m = \frac{0.06 \cdot 400K \cdot e^{0.06 \cdot 30}}{e^{0.06 \cdot 30} - 1} \approx 28.75 K/\text{year}.$$

$m/12 \approx \$2400/\text{month}$.

Recall that the graph of the solution looks like this: $\longrightarrow$



See Ex. 7 in Lec. 10.

(c)

Total amount paid to the bank =
$m \cdot T \approx 862.5\ K$.

7 The logistic growth model

A typical application leading to this model would be:

A disease spreads in a population of N people at a rate proportional to the product of those who have the disease and those who have not yet had it. State the corresponding DE.

$$\underbrace{\frac{dP}{dt}}_{\substack{\text{rate of change of} \\ \text{\# of people } P \text{ who} \\ \text{have the disease}}} = \overbrace{K \cdot}^{\text{proportional to}} \underbrace{P}_{\substack{\text{those who} \\ \text{have the} \\ \text{disease}}} \cdot (N - \underbrace{P)}_{\substack{\text{those who} \\ \text{do not have it}}}$$

This DE is separable. Its solution is more complicated than the simpler exponential solutions of the other models that we've considered. It is listed in the textbook. In these Notes it will suffice to present representative plots of this solution for different initial conditions P_1, P_2, P_3. The lower curve is a typical **logistic growth curve.**