Lecture 12 — Solution of linear DEs, with application to Mixing Problems

① Introduction and definition

The four types of DEs

$$y' = f(x, y) \qquad (1)$$

whose analytical solutions can be found systematically, are:

- separable (Lecture 11);
- linear (this Lecture);
- so-called "exact" — some generalization of separable (not considered in this course);
- and any other that can be transformed to one of the above three types (also not considered in this course).

Linear DEs have this form:

$$\boxed{y' = p(x)\cdot y + g(x)} \qquad (2)$$

for any $p(x), g(x)$ (which **depend** **only** **on** x but **not** **on** y).

Note: Notations of this Lecture differ from those of Lec. 9.3 because I wanted to stay with the basic concept and notations stated in Lec. 10: Solution of any DE begins by putting it in the form of Eq. (1). Therefore, the notation $f(\dots)$ is reserved for the r.h.s. of (1). Clearly, in Eq. (2), $f(x,y) \equiv p(x)\cdot y + g(x)$.

Ex. 1(a) $\quad xy' + \sin x \cdot y = x^3$

is a linear DE. Indeed, first put it in the form of Eq. (1):

$$xy' = -\sin x \cdot y + x^3 \Rightarrow y' = \underbrace{-\frac{\sin x}{x}}_{\text{this is } p(x)} \cdot y + \underbrace{x^2}_{g(x)}$$

Ex. 1(b) $\quad y' = x + y^2$ is not a linear DE:

$$y' = \underbrace{y}_{} \cdot y + x$$

↑ this is not $p(x)$,

so this equation is not of the form of Eq. (2).

② Algorithm for solving linear DEs (2)

1) Compute $\mathcal{P}(x) = \int p(x)\,dx$ (ignore the "+C" at this step);
2) Compute $e^{\mathcal{P}(x)}$.
3) Compute $e^{-\mathcal{P}(x)} = \frac{1}{e^{\mathcal{P}(x)}} \equiv I(x)$

   This $I(x)$ is called the "integrating factor".
4) The solution of $\quad y' = p(x) \cdot y + g(x)$ is given by:

$$y(x) = e^{\mathcal{P}(x)} \cdot \Big(\underbrace{\int^x I(s)\, g(s)\, ds}_{\text{Include the "+C" at this step.}} \Big)$$

See the Notes about the $\int^x$-integral below.

Notes about the notation $\int^{x} I(s)g(s)\,ds$:

1. Notation $\int^{x} I(s)g(s)\,ds$ simply means this: Compute the integral and then set $s = x$.
2. This is more literate than writing $\int I(x)g(x)\,dx$, although both lead to the same result. The reason why I do not use the latter notation is because its integration variable (which is a dummy variable and runs over a range of values) is named the same as the argument of $y(x)$, which (the "x" on the l.h.s.) has some one fixed value.
3. This notation in the formula $e^{P(x)} \int^{x} e^{-P(s)} g(s)\,ds$ should also serve to suppress the temptation to cancel out $e^{P(x)}$ outside the integral with $e^{-P(s)}$ inside it (see Ex. 2 below).
4. Finally, as with any indefinite integral, do not forget the "+C" inside the parentheses.

Ex. 2 Find the general solution of this linear DE:

$$y' = -\frac{2}{x+3} \cdot y + (x+3) \quad (\text{assume } x+3 > 0)$$

Sol'n: Follow the algorithm.

1) $p(x) = -\dfrac{2}{x+3} \Rightarrow$

$$P(x) = -2\int \frac{dx}{x+3} \overset{\boxed{u = x+3,\ du = dx}}{=} -2\int \frac{du}{u} = -2\ln|x+3| \overset{\text{since } (x+3)>0}{=} -2\ln(x+3).$$

$\boxed{\text{Appendix A.5, Thm. 1}}$

2) $e^{\mathcal{L}(x)} = e^{-2\ln(x+3)} \overset{\downarrow}{=} \left(e^{\ln(x+3)}\right)^{-2}$

$= (x+3)^{-2} \equiv \dfrac{1}{(x+3)^2}$

3) $I(x) = \dfrac{1}{e^{\mathcal{L}(x)}} = \dfrac{1}{(x+3)^{-2}} = (x+3)^2$

4) a) $\int^x I(s)g(s)\,ds = \int^x (s+3)^2\cdot(s+3)\,ds = \left| \begin{array}{l} u = s+3 \\ du = ds \end{array} \right.$

$= \int^{s=x} u^2\cdot u\,du = \int^{s=x} u^3\,du = \dfrac{u^4}{4} + C = \dfrac{(x+3)^4}{4} + C$

b) $y(x) = \underbrace{(x+3)^{-2}}_{e^{\mathcal{L}(x)}} \cdot \left(\dfrac{(x+3)^4}{4} + C\right) = \dfrac{(x+3)^2}{4} + \dfrac{C}{(x+3)^2}$

Notes: $\boxed{1}$ You **absolutely must** feel comfortable with the algebra in Step 2.

Again, review Appendix A.5, Thm. 1, if you do not.

$\boxed{2}$ Notice what would have happened if you had written

$e^{\mathcal{L}(x)} \int e^{-\mathcal{L}(x)} g(x)\,dx = \dfrac{1}{(x+3)^2} \int (x+3)^2\cdot(x+3)\,dx$

and had cancelled out $(x+3)^2$ outside and inside the integral. Then you would have obtained:

WRONG WAY, DO NOT FOLLOW: $\cancel{\dfrac{1}{(x+3)^2}} \int \cancel{(x+3)^2}\cdot(x+3)\,dx$

$= \int (x+3)\,dx = \dfrac{(x+3)^2}{2} + C,$

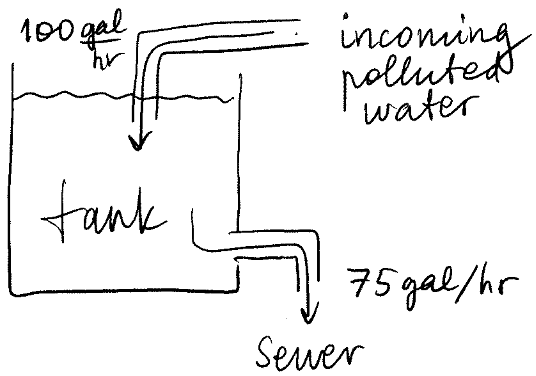
which is different than the correct answer found above.

MORAL (again): Never cancel $e^{\mathcal{L}(x)}$ (or $\frac{1}{I(x)}$) outside the $\int$ with the $e^{-\mathcal{L}(x)}$ (or $I(x)$) inside the $\int$.

③ Application to mixing problems

Ex. 3 (= slightly modified Ex. 9.3.5, with different order of steps and different notations)

A tank is used to control the release of pollutants into a sewage system: polluted water comes into the tank, gets thoroughly mixed and then released into a sewer.



Initially, the tank contains 500 gal of water with 2 lb of pollutants per gallon. Polluted water containing 5 lb of pollutants per gallon is pumped into the tank at the rate of 100 gal/hr. Simultaneously, the thoroughly mixed contents of the tank are released into a sewer at the rate of 75 gal/hr. At the end of 5 hrs, determine:

a) The total amount of pollutants in the tank.

b) The **concentration** (in lb/gal) at which pollutants are being released into a sewer.

(Note: the textbook uses the word "rate" instead of "concentration". This is incorrect. Please use "concentration.")

Sol'n: 1) You always begin by writing an equation for the rate of change of the amount, **W**, of water:

$$\frac{dW}{dt} = \left(\begin{matrix}\text{Flow of} \\ \text{water } \textbf{in}\end{matrix}\right) - \left(\begin{matrix}\text{Flow of} \\ \text{water } \textbf{out}\end{matrix}\right)$$

$$\frac{dw}{dt} = 100 - 75 = 25 \Rightarrow$$

$$w = \int 25\,dt = 25t + C.$$

$$w(0) = 500 \Rightarrow 25 \cdot 0 + C = 500 \Rightarrow C = 500 \Rightarrow \boxed{w = 25t + 500}$$

2) Write a similar DE for the rate of change of the amount of pollutants.

(Note: Call this amount $m(t)$. Do not use the book's notation $p(t)$ since we are using $p(t)$ for a coefficient in the DE.)

$$\frac{dm}{dt} = \begin{pmatrix}\text{Flow of} \\ \text{pollutants in}\end{pmatrix} - \begin{pmatrix}\text{Flow of} \\ \text{pollutants out}\end{pmatrix}$$

- $$\begin{pmatrix}\text{Flow of} \\ \textbf{pollutants in}\end{pmatrix} = \begin{pmatrix}\text{Flow of} \\ \textbf{water in}\end{pmatrix} \cdot \begin{pmatrix}\text{concentration of} \\ \textbf{incoming pollutants}\end{pmatrix}$$

$$= 100 \frac{\cancel{gal}}{hr} \cdot 5 \frac{lb}{\cancel{gal}} = 500 \frac{lb}{hr}$$

- $$\begin{pmatrix}\text{Flow of} \\ \textbf{pollutants out}\end{pmatrix} = \begin{pmatrix}\text{Flow of} \\ \textbf{water out}\end{pmatrix} \cdot \begin{pmatrix}\text{concentration of} \\ \textbf{outgoing pollutants}\end{pmatrix}$$

$$= 75 \cdot \frac{(\text{amount of pollutants in tank} = m(t))}{(\text{amount of water in tank} = w(t))}$$

$$= 75 \cdot \frac{m \leftarrow \boxed{\textbf{still unknown!}}}{25t + 500}$$

- Technical simplification of the last term:
It will be useful to make the coefficient of t in the denominator equal 1. For that, divide both the numerator and denominator by 25:

$$\frac{75m}{25t+500} = \frac{75m/25}{(25t+500)/25} = \frac{3m}{t+20}$$

■ Put everything together into the DE:

$$\frac{dm}{dt} = 500 - \frac{3m}{t+20} \Rightarrow \frac{dm}{dt} = \underbrace{-\frac{3}{t+20}}_{p(t)} \cdot m + \underbrace{500}_{g(t)}$$

This is a linear DE!

3) Find its general solution following The Algoritm:

$\boxed{1}$ $\underline{P}(t) = \int p(t)\,dt = \int \frac{-3}{t+20}\,dt \underset{\substack{\uparrow \\ \text{similar} \\ \text{to Ex. 2}}}{=} -3\ln(t+20)$

$\boxed{2}$ $e^{P(t)} = e^{-3\ln(t+20)} = \left(e^{\ln(t+20)}\right)^{-3} = (t+20)^{-3}$

$\boxed{3}$ $I(t) = \frac{1}{e^{P(t)}} = (t+20)^3$

$\boxed{4}$ a) $\int^t I(s) g(s)\,ds = \int^t (s+20)^3 \cdot 500\,ds =$ $\left|\begin{array}{l} s+20=u \\ ds = du \end{array}\right.$

$= 500 \int^{s=t} u^3\,du = 500 \cdot \frac{u^4}{4}\Big|^{s=t} + C = 125(t+20)^4 + C$

b) $m(t) = (t+20)^{-3}\left(125(t+20)^4 + C\right) = 125(t+20) + \frac{C}{(t+20)^3}$

4) Find the particular solution satisfying the given $m(0)$.

a) Initially, there are 500 gal of water with 2 lb/gal of pollutants, so: $m(0) = 500 \cdot 2 = 1000$ lb.

b) Solve for C:

$$1000 = m(0) = 125(0+20) + \frac{C}{(0+20)^3}$$

Step 3) above

$$1000 = 2500 + \frac{C}{20^3} \Rightarrow C = -20^3 \cdot 1500$$ (Can multiply out, but see next step.)

c) Substitute C back into $m(t)$:

$$m(t) = 125(t+20) - \frac{1500 \cdot 20^3}{(t+20)^3}.$$

again, Thm. 1 of App. A.5

Simplify the last term: $$\frac{1500 \cdot 20^3/20^3}{(t+20)^3/20^3} = \frac{1500}{((t+20)/20)^3}$$

$$= \frac{1500}{(\frac{t}{20}+1)^3} = \frac{1500}{(1+t/20)^3}$$

So: $$\boxed{m(t) = 125(t+20) - \frac{1500}{(1+t/20)^3}}$$

5) $m(5) = 125(5+20) - \frac{1500}{(1.25)^3} = 2357$ lb.

6) The concentration at which pollutants leave the tank is $(m(t)/w(t))$ – see Step 2) on p. 12-6.

Thus: $$\frac{m(5)}{w(5)} = \frac{2357}{25 \cdot 5 + 500} = 3.77 \frac{\text{lb}}{\text{gal}}$$