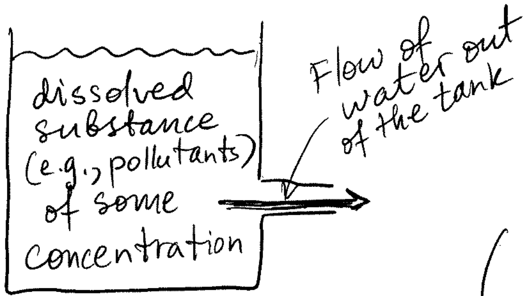
# Lecture 13 - Functions of several variables and visualization of functions of two variables

## ① Motivation and examples

In the last example in Lecture 12 we used a formula for the rate of flow of pollutants in or out of a tank:




$$\begin{pmatrix}\text{Flow rate of a}\\ \text{dissolved substance}\\ \text{out of the tank}\end{pmatrix} = \begin{pmatrix}\text{Flow rate of}\\ \text{water}\\ \text{out of the tank}\end{pmatrix} \cdot \begin{pmatrix}\text{concentration of}\\ \text{the dissolved substance}\\ \text{in water}\end{pmatrix}$$

So:

$$\underbrace{F_{\text{substance}}}_{\text{Flow rate}} = \underbrace{F_{\text{water}}} \cdot \underbrace{C_{\text{substance}}}_{\text{concentration}}$$

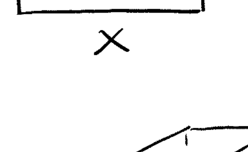
This depends on 2 variables:

The are infinitely many examples where the quantity of interest depends on several input variables.

### Ex. 1 Examples of functions of several variables which will be used in later lectures.

**Ex. 1(a)** The area of a rectangle is:

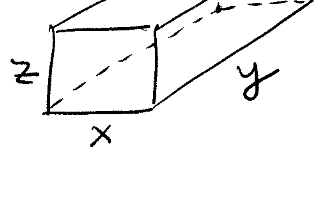
$$A(x,y) = x \cdot y.$$




So, e.g., $A(11,12) = 11 \cdot 12 = 132$ (sq. units)

**Ex. 1(b)** The volume of a rectangular box:

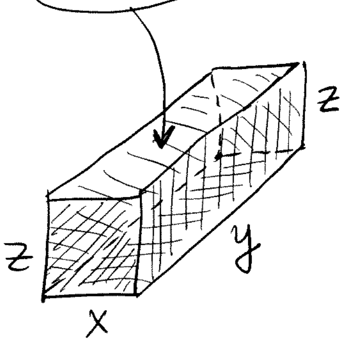
$$V(x,y,z) = x \cdot y \cdot z$$



## Ex. 1(c) Surface area of a box with an open top

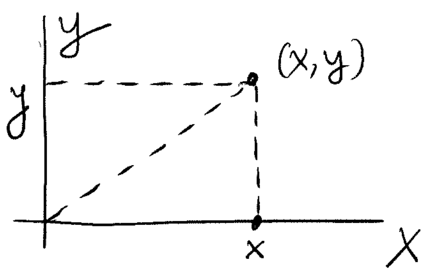
$$A(x,y,z) = A_{bottom} + (A_{front} + A_{back}) + (A_{\text{left face}} + A_{\text{right face}})$$

$$= x\cdot y + x\cdot z\cdot 2 + y\cdot z\cdot 2$$

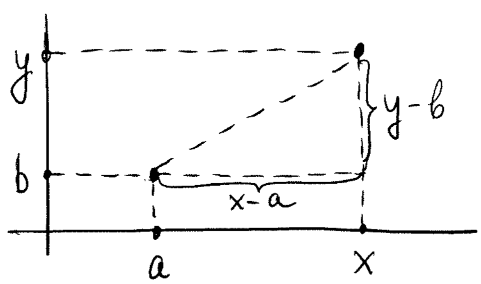



$A(10,11,12) = 10\cdot 11 + 10\cdot 12\cdot 2 + 11\cdot 12\cdot 2 = 614$ (sq. units)

## Ex. 1(d) Distance between two points




Recall that the distance between $(0,0)$ = "the origin" and point $(x,y)$ is $\sqrt{x^2+y^2}$; see the figure.
(Note: Excuse abuse of notations: $(x,y)$ denote coordinates of the point but also label the axes.)

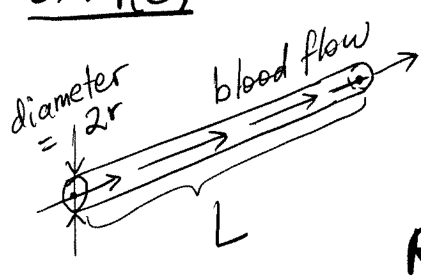



New: Distance between points $(a,b)$ & $(x,y)$:

$$D(a,b,x,y) = \sqrt{(x-a)^2 + (y-b)^2}$$

$$D(2,3,5,7) = \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{9+16} = 5.$$

## Ex. 1(e)




When liquid flows in a cylindrical tube, it experiences resistance. Think of a blood flow in vessels.

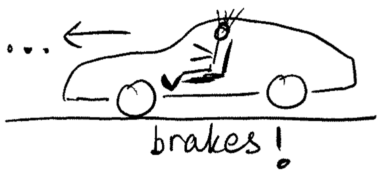
$$R(L,r) = \underbrace{k\cdot}_{\text{"proportional to"}} L/r^4,$$

resistance → $R$

$L$ = length, $r$ = radius of vessel

Heart provides pressure to overcome this resistance.

## Ex. 1(f)




When a car driver slams on the brakes, the length of the skid to a stop is given by the formula:

$$L(w,v) = \underbrace{k}_{\text{proport. to}} \cdot \underbrace{w}_{\text{weight}} \cdot \underbrace{v^2}_{\text{speed}}$$

By doubling the weight of the car on doubles $L$, but by going twice as fast one quadruples $L$!

## Ex. 1(g) Profit of two competing products

The store sells two competing products:

**A at price \$p/unit** and **B at price \$q/unit.**

The store **pays a supplier** \$60/unit for A and \$80/unit for B.

The **demands** are $x$ units for A and $y$ units for B, so that:

$$x = 260 - 3p + q$$
$$y = 180 + p - 2q$$

Find the Revenue, Cost, and Profit for A and B.

$$\text{Revenue}(p,q) = \underbrace{x}_{\#\text{ of units of A}} \cdot \underbrace{p}_{\text{price per unit}} + \underbrace{y \cdot q}_{\text{similarly for B}}$$

$$= (260 - 3p + q) \cdot p + (180 + p - 2q) \cdot q$$

$$\text{Cost}(p,q) = x \cdot 60 + y \cdot 80 = (260 - 3p + q) \cdot 60 + (180 + p - 2q) \cdot 80$$

$$\text{Profit}(p,q) = \text{Revenue}(p,q) - \text{Cost}(p,q)$$

$$= (260 - 3p + q) \cdot (p - 60) + (180 + p - 2q) \cdot (q - 80).$$

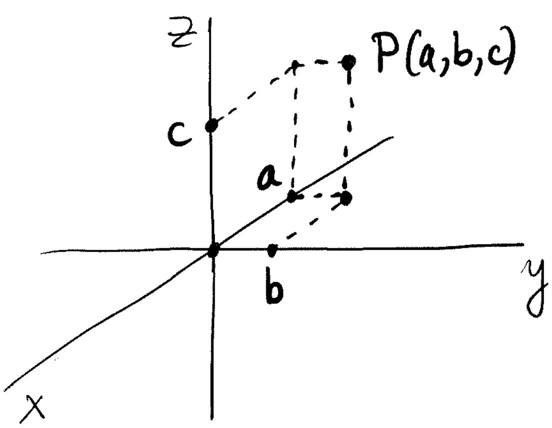
## ② Visualizing functions of two variables

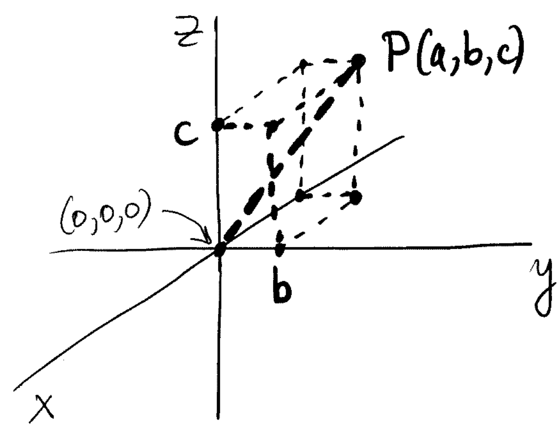
### 2a Idea

To visualize a function $f(x)$ of 1 variable, one plots a **curve** $y = f(x)$ in the $(x,y)$-plane.

To visualize a function $f(x,y)$ of 2 variables, one plots a **surface** $z = f(x,y)$ in the $(x,y,z)$-space.

### 2b The $(x,y,z)$-space (a.k.a. the 3D coordinate system)



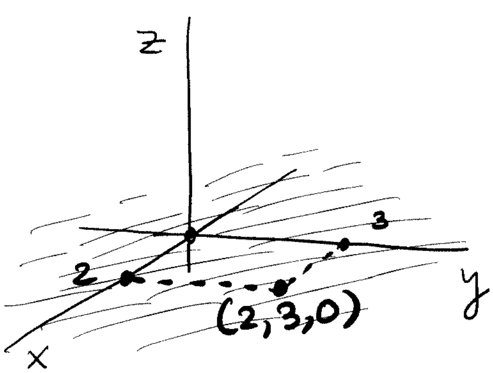




$a<0,\ b>0,\ c>0$

Think of the rectangular box with sides $a, b, c$, with the origin $(0,0,0)$ at one corner and with $P(a,b,c)$ in the farthest corner from it.

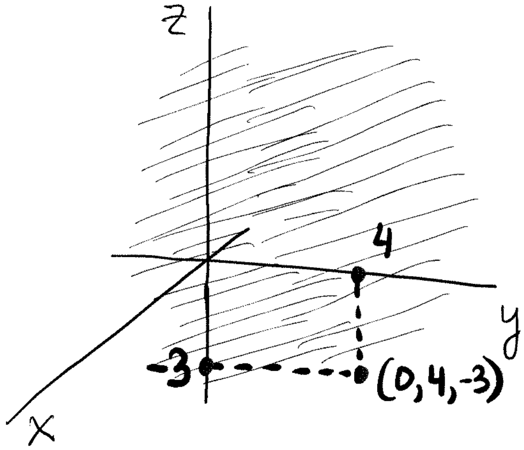
We will consider some representative surfaces soon.

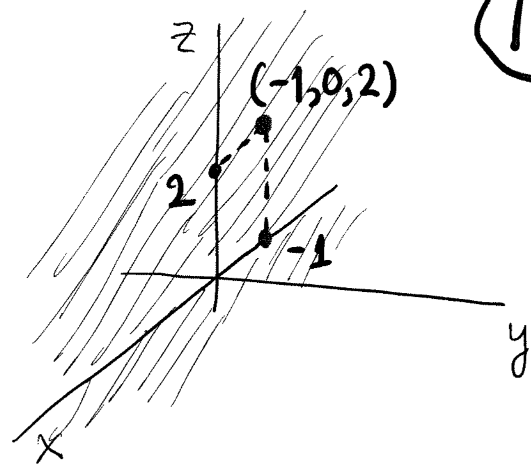
### 2c Coordinate planes




$\boxed{z=0}$ a.k.a. the $\boxed{\text{xy-plane}}$

In this plane, $x$ & $y$ can have any values, and the elevation $z$ is zero.
Think of the floor in a room.

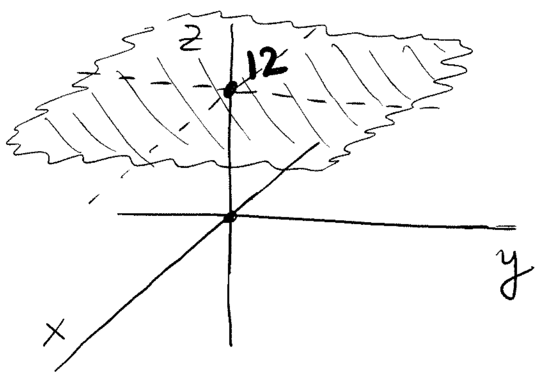


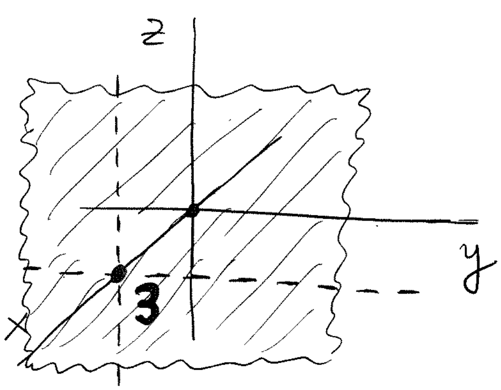

$\boxed{x=0}$ a.k.a. $\boxed{yz\text{-plane}}$
is the plane where $x=0$
and $y, z$ can have any values.
(The "back wall" of a room.)

$\boxed{y=0}$ a.k.a. $\boxed{xz\text{-plane}}$
is the plane where $y=0$
and $x, z$ can have any values.
(The "left wall" of a room.)

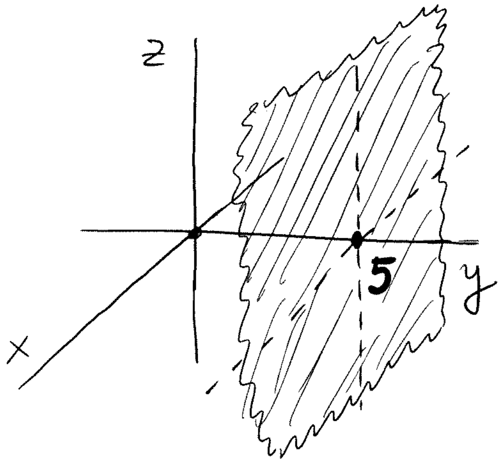
## 2d Planes parallel to coordinate planes




$\boxed{z=c}$, e.g., $z=12$,
is a **horizontal plane** || **xy-plane**
**c units** away from that plane.
$z=12$ ⇒ think of a 12-foot ceiling in a room.



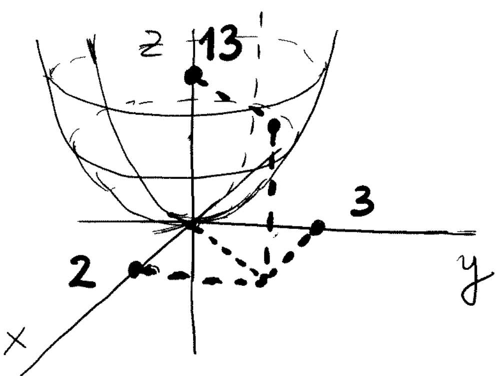

$\boxed{x=a}$, e.g., $x=3$,
is a **vertical plane** || **yz-plane**
**a units** away from that plane.
$x=3$ is a "front wall" of a room that is 3 units away from the back wall.




$\boxed{y=b}$, e.g., $y=5$, is a **vertical plane** $\parallel$ **xz-plane** that is **b units** away from that plane.
$y=5$ is the "right wall" of a room 5 units away from the "left wall".

## $\boxed{2e}$ <u>2.5 types of important surfaces</u>

"<u>Type 1</u>": $\boxed{z=f(x,y)=x^2+y^2}$



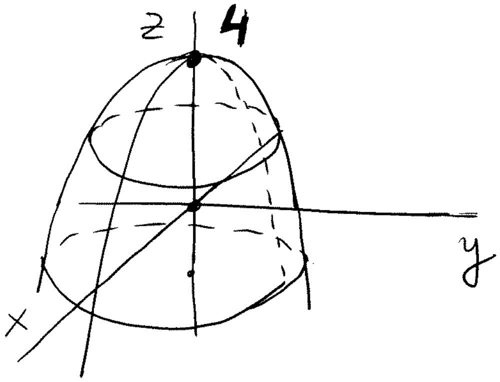

This is called a "circular **paraboloid**" (the shape of a bowl). It has the **minimum** at $(x,y,z)=(0,0,0)$.

E.g., $f(2,3)=2^2+3^2=13$

"<u>Type 1.5</u>": **Upside-down** circular **paraboloid**

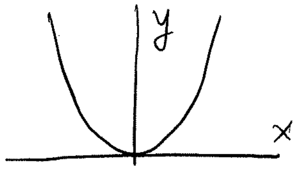
$$z=\underset{\uparrow}{4}-x^2-y^2$$

**can be any number**

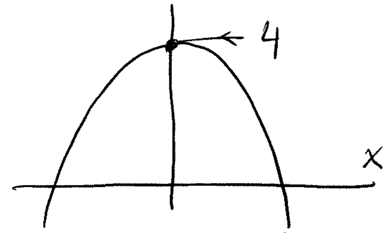



It has a **maximum** at $(4,0,0)$.

The above two surfaces have similarities to functions of 1 variable:



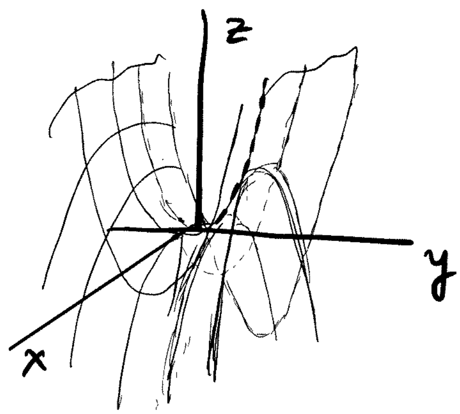

and




$y = x^2$, minimum @ $(0,0)$ $\qquad$ $y = 4 - x^2$, maximum @ $(0,0)$

The **next** type of **surface** is a **hybrid of these two** and does **not** have a counterpart among functions of one variable.

## "Type 2.5" : Saddle surface (a hyperbolic paraboloid)

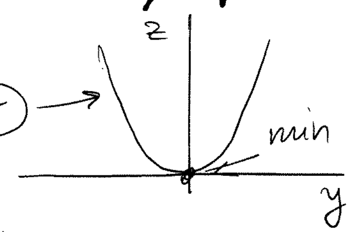



$$z = y^2 - x^2$$

($z = x^2 - y^2$ is obtained by a $90^\circ$ rotation)

It has a **minimum** in the **yz-plane**:

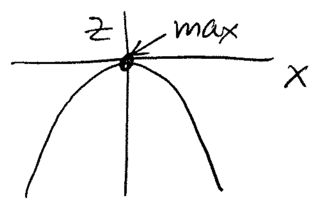
$z = y^2 - 0^2 = y^2$ : $\quad$ $x=0$ . $\quad$ $z = y^2$




but a **maximum** in the **xz-plane**:

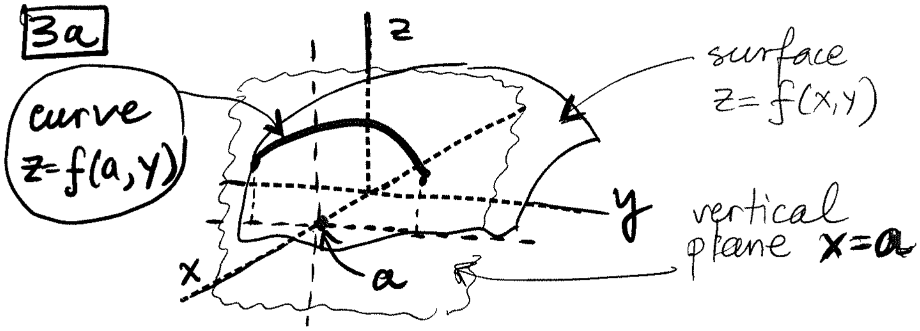
$z = 0 - x^2 \Rightarrow z = -x^2$.

$y=0$




## ③ Cross-sections of surfaces with planes $x = \text{const}$, $y = \text{const}$, $z = \text{const}$.

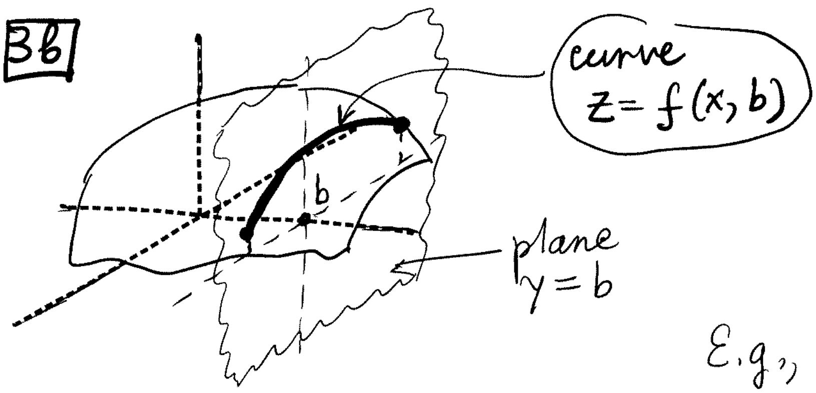
$\boxed{3a}$




Cut the **surface** $z = f(x,y)$ with plane $x = a$:

You get a curve $z = f(a, y)$.
E.g., surface $z = 9 - x^2 - y^2$ cut with plane $x = 1$ produces a curve $z = 9 - 1^2 - y^2 = 8 - y^2$.

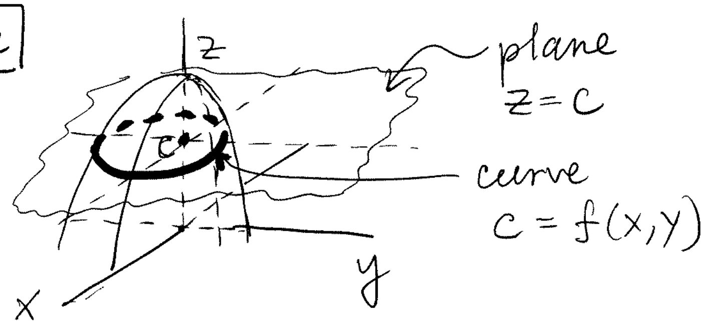
**3b**




Cut the surface $z = f(x, y)$ with plane $y = b \Rightarrow$ get a curve $z = f(x, b)$.

E.g., surface $z = 9 - x^2 - y^2$ cut with plane $y = 2$ produces a curve $z = 9 - x^2 - 2^2 = 5 - x^2$.

**3c**




E.g., $z = 9 - x^2 - y^2$ cut with $z = 5$ produces:
$5 = 9 - x^2 - y^2 \Rightarrow$
$x^2 + y^2 = 9 - 5 = 4$,
which is a circle of radius 2.

Note: A collection of curves obtained by cutting a 3D map of a terrain by horizontal planes at equal elevation levels, e.g., $z = 1$, $z = 2$, $z = 3$, etc. is called a topographic map.
Similarly for the temperature map (on a weather channel, etc.).