Lecture 14 — Partial derivatives

① Definition of (first-order) partial derivatives and their visualization for functions of 2 variables

1a Definition

Consider some function $z = f(x,y)$. Suppose one fixes y at some value (keep calling it "y") and takes the derivative with respect to (w.r.t.) x:

$$\left.\frac{df}{dx}\right|_{y=\text{const}}.$$

This is called the **partial derivative of f w.r.t. x** and denoted:

$$\left.\frac{df}{dx}\right|_{y=\text{const}} \equiv \frac{\partial f}{\partial x} \leftarrow \left(\text{"partial } f \text{ partial } x\text{"}\right) \text{ or, also: } f_x \quad \left(\text{"}f \text{ sub-}x\text{"}\right)$$

Similarly, the **partial derivative of f w.r.t. y** is:

$$\left.\frac{df}{dy}\right|_{x=\text{const}} \equiv \frac{\partial f}{\partial y} \quad (\text{say it: } \ldots) \text{ or, also: } f_y \quad (\text{say it: } \ldots)$$

Ex. 1 For $f(x,y) = 5 - x^2 + 2y^2 - 3x^2y + 4xy^2$ find:
(a) f_x, (b) f_y, (c) $f_x(1,2)$, (d) $f_y(1,2)$.

Sol'n:

(a) $f_x = (5 - x^2 + 2y^2 - 3x^2y + 4xy^2)_x = 0 - 2x + 0 - 6xy + 4y^2$
$= -2x - 6xy + 4y^2$

(b) $f_y = (5 - x^2 + 2y^2 - 3x^2y + 4xy^2)_y = 0 - 0 + 4y - 3x^2 + 8xy$
$= 4y - 3x^2 + 8xy$

(c) To find $f_x(1,2)$, take the expression for f_x and substitute $x=1, y=2$. Never substitute first and then differentiate!

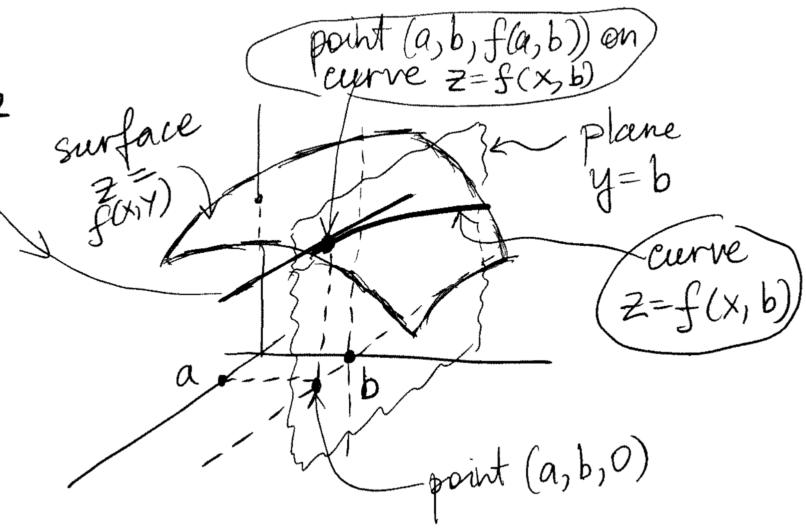
$f_x(1,2) = -2\cdot 1 - 6\cdot 1\cdot 2 + 4\cdot 2^2 = 2$

(d) $f_y(1,2) = 4\cdot 2 - 3\cdot 1^2 + 8\cdot 1\cdot 2 = 21$

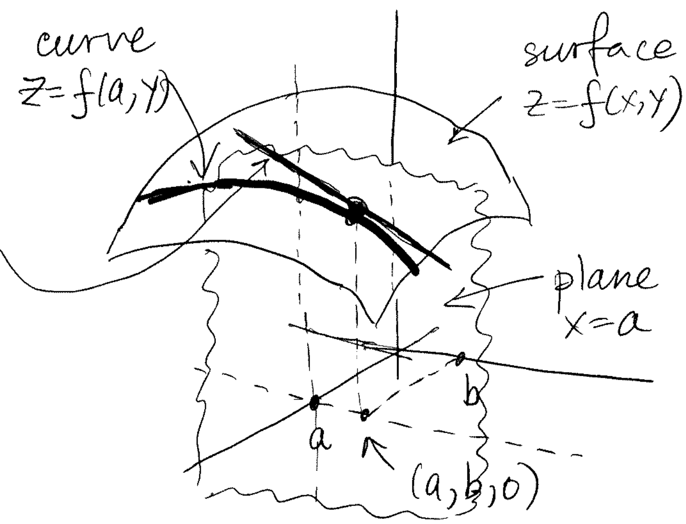
1b Visualization of $f_x(x,y)$ and $f_y(x,y)$

This is the **tangent line** to curve $z=f(x,b)$ at point $(a,b,\underbrace{f(a,b)}_{=z})$.

$\boxed{\text{Its slope equals } f_x(a,b).}$

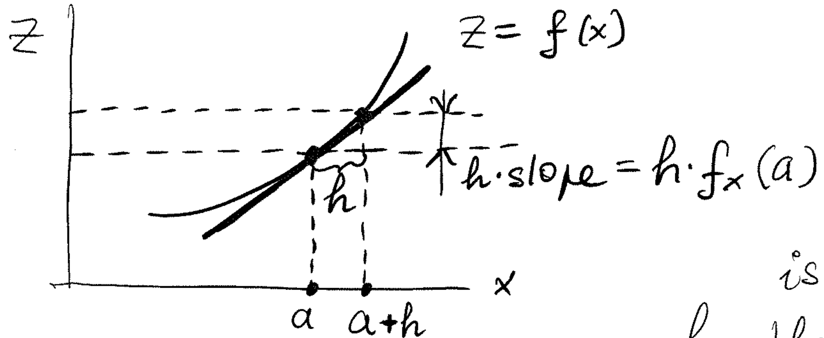


Similarly,
$f_y(a,b)$ is the slope of the tangent line to curve $z=f(a,y)$ at point $(a,b,\underbrace{f(a,b)}_{z})$



(2) Partial derivatives as rates of change, and Applications

Since partial derivatives are slopes of curves, we can explain the main idea of this topic using the graph of $z=f(x)$. (The notation "y" is reserved for the second variable. You can think of this curve as $z=f(x,b)$ — a fixed value of y.)



The actual change of the curve $z = f(x)$ is closely approximated by the change measured along the tangent line.

So:

$\boxed{\text{$h \cdot f_x(a,b)$ approximates the change of $f(x,y)$ when $y = b$ is kept constant and x changes from a to $a+h$}}$

Similarly:

$\boxed{h \cdot f_y(a,b)}$ ... <say it to practice>

Often (but not always!) the "change" h of the variable can be taken as "1 unit".

Ex. 2 In Ex. 1(g) of Lec. 13 we considered the Revenue, Cost, and Profit of two competing products A and B:

$x = 260 - 3p + q$ $\qquad$ $y = 180 + p - 2q$

x: (demand for A in # of units)
p: price of 1 unit of A
q: price of 1 unit of B
y: (demand for B, in # of units)

$R(p,q) = x \cdot p + y \cdot q = (260 - 3p + q) \cdot p + (180 + p - 2q) \cdot q$
$\quad = 260p - 3p^2 + 2pq + 180q - 2q^2$

↑ revenue

$C(p,q) = x \cdot 60 + y \cdot 80 = (260 - 3p + q) \cdot 60 + (180 + p - 2q) \cdot 80$
$\quad = 20{,}000 - 100p - 100q$

↑ cost

$$P(p,q) = R(p,q) - C(p,q) =$$
$\uparrow$ profit
$$= 360p - 3p^2 + 2pq + 280q - 2q^2 - 20{,}000$$

Find $P_p(15,25)$ and $P_q(15,25)$ **and interpret the results.**

Sol'n: 1) $P_p(p,q) = 360 - 6p + 2q + 0 - 0 - 0 = 360 - 6p + 2q$

$P_q(p,q) = 0 - 0 + 2p + 280 - 4q - 0 = 280 + 2p - 4q$

2) $P_p(15,25) = 360 - 6 \cdot 15 + 2 \cdot 25 = 320$

$P_q(15,25) = 280 + 2 \cdot 15 - 4 \cdot 25 = 210$

3) Interpretation

Just before this Example we stated that

$h \cdot f_x(a,b)$ approximates the change of $f(x,y)$ when $y = b$ is kept constant and x changes from a to $a+h$.

In the notations of this Example:

$$\underset{\text{"}f\text{"}}{P}\,{}_{\underset{\text{"}x\text{"}}{p}}\,(\underset{\text{"}a\text{"}}{15}, \underset{\text{"}b\text{"}}{25})$$

We also **need some meaningful value for h.**

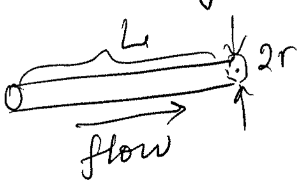
Since our variables are p & q — prices of the products in units of \$, — then it is meaningful to take $h = \$1$ (as opposed to cents, thousands of \$, or euros). Then the general statement in the box becomes:

$P_p(15,25) = 320$ means that by increasing the price of A (per unit) from \$15 to \$16 while keeping the price of B at \$25, one will increase (since $320 > 0$) the profit by \$320.

Similarly, $P_q(15,25) = 210$ means that by keeping the price of A at \$15 but increasing the price of B from \$25 to \$26, one will increase (since $210>0$) the profit by \$210.

Ex. 3 In Ex. 1(e) of Lec. 13 we had Poiseuille's formula for the resistance of a viscous flow in a capillary:

$$R(L,r) = K \cdot \frac{L}{r^4}$$



Suppose that in some units, $K=1$. Find $R_L(3, 0.5)$ and $R_r(3, 0.5)$ and interpret the results.

Sol'n: 1) $R_L(L,r) = \cancel{K}^{1} \cdot \frac{1}{r^4} = 1/r^4$

$R_r(L,r) = (L \cdot r^{-4})_r = L \cdot (-4) \cdot r^{-5} = -4L/r^5$

2) $R_L(3, 0.5) = 1/0.5^4 = 16$

$R_r(3, 0.5) = -4 \cdot 3/0.5^5 = -384$

3) Interpretation

We begin by finding a meaningful value for h, which will denote a change in L and, separately, a change in r.

If $L = 3$ (length units), it is not a very good idea to take $h = 1$ (same length unit), because the change h is implied to be small in some sense, and "1" is not small compared to "3".

Moreover, given that $r = 0.5$ (length units), it would be meaningless to take $h = 1$ (length units). Indeed, not only is $(h=1) > (r=0.5)$, but also, if we decrease r by h instead of increasing it, we will get a negative radius: $0.5 - 1 = -0.5$, which is, again, meaningless.

In such situations, where using $h = 1$ unit is not a meaningful choice, a **good alternative is to take $h = 1\%$ of the variable that is changing.**

a) To interpret $R_L(3, 0.5)$, choose $h = 1\% \cdot L$ (it is more illuminating to get the answer for general L and r and then substitute $L = 3$ and $r = 0.5$).

Then $h \cdot R_L(L, r) = \underbrace{(1\% \cdot L)}_{h} \cdot \underbrace{\left(\frac{1}{r^4}\right)}_{R_L} = 1\% \cdot \frac{L}{r^4} = 1\% \cdot R$

$\boxed{\text{By increasing } L \text{ by } 1\% \text{ and keeping } r \text{ fixed, one will increase } R \text{ by } 1\%.}$

b) To interpret $R_r(L, r)$, choose $h = 1\% \cdot r$

Then $h \cdot R_r(L, r) = 1\% \cdot r \cdot \left(-4 \frac{L}{r^5}\right) =$

$= -4\% \cdot \cancel{r} \cdot \frac{L}{r^{5-1}} = -4\% \cdot \frac{L}{r^4} = -4\% \cdot R$

Thus:

$\boxed{\text{By increasing } r \text{ by } 1\% \text{ and keeping } L \text{ fixed, one will decrease } R \text{ by } 4\%.}$

(You do not need to substitute $L = 3$, $r = 0.5$.)

Moral: When the problem asks for an interpretation of the results, put effort into deciding **whether to take** $h = 1$ unit **or** 1% **of the variable.**

Rule of thumb: Take $h = 1\%$ **of the variable when** $f = k \cdot x^m \cdot y^n$ and take $h = 1$ **unit in all other cases.**

③ Second-order partial derivatives

These are defined similarly to $d^2 f(x)/dx^2$:

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv (f_x)_x \equiv f_{xx}$$

Similar for $\partial^2 f/\partial y^2 \equiv f_{yy}$.

Mixed partial derivative:

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv (f_y)_x \equiv f_{yx}$$

Note that according to these notations, the order of x & y is opposite.

Similarly:

$$\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv (f_x)_y \equiv f_{xy}.$$

As you can see, keeping track of the order of x & y in these notations can easily become confusing.

Fortunately, for all functions considered in this course, we will have: $\boxed{f_{xy} = f_{yx}}$ **!!!**

So, it will not matter in which order you need to differentiate.

<u>Sidenote:</u> There exist functions for which $f_{xy} \neq f_{yx}$. They are <u>not</u> pathological, and they have important applications. However, we will not consider them in this course.

<u>Ex. 4</u> For $f(x,y) = x^2y^3 - \frac{x}{y^2} + e^{2x^2y}$, compute f_{xx}, f_{yy}, f_{xy}, f_{yx}.

1) Need to compute f_x, f_y first.

$u = 2x^2y$

$$f_x = \left(x^2y^3 - \frac{x}{y^2} + e^{2x^2y}\right)_x$$
$$= 2x \cdot y^3 - \frac{1}{y^2} + \underbrace{\frac{de^u}{du}}_{e^u} \cdot \underbrace{\frac{\partial u}{\partial x}}_{4x \cdot y}$$
$$= 2xy^3 - \frac{1}{y^2} + e^{2x^2y} \cdot 4xy$$

$$f_y = \left(x^2y^3 - \frac{x}{y^2} + e^{2x^2y}\right)_y$$
$$= x^2 \cdot 3y^2 - x \cdot (y^{-2})' + \frac{de^u}{du} \cdot \underbrace{\frac{\partial u}{\partial y}}_{2x^2}$$
$$= x^2 \cdot 3y^2 + 2x/y^3 + e^{2x^2y} \cdot 2x^2.$$

2) $$f_{xx} = (f_x)_x = \left(2xy^3 - \frac{1}{y^2} + e^{2x^2y} \cdot 4xy\right)_x$$
$$= 2y^3 - 0 + \underbrace{\left(\left(e^{2x^2y}\right)_x \cdot 4xy + e^{2x^2y} \cdot (4xy)_x\right)}_{\text{product rule}}$$
$$= 2y^3 + \left(\underbrace{e^{2x^2y} \cdot 4xy}_{\text{did in } f_x} \cdot 4xy + e^{2x^2y} \cdot 4y\right)$$
$$= 2y^3 + e^{2x^2y}\left(16x^2y^2 + 4y\right)$$

$$f_{yy} = (f_y)_y = \left(3x^2y^2 + 2x \cdot y^{-3} + e^{2x^2y} \cdot 2x^2\right)_y$$
$$= 3x^2 \cdot 2y + 2x \cdot (-3y^{-4}) + \underbrace{\left(e^{2x^2y}\right)_y}_{\text{did in } f_y} \cdot 2x^2$$
$$= 6x^2y - 6xy^{-4} + e^{2x^2y} \cdot \underbrace{2x^2 \cdot 2x^2}_{4x^4}$$

$$f_{xy} = (f_x)_y = \left(2xy^3 - y^{-2} + e^{2x^2y}\cdot 4xy\right)_y$$

$$= 2x\cdot 3y^2 + 2y^{-3} + \underbrace{\left((e^{2x^2y})_y\cdot 4xy + e^{2x^2y}\cdot(4xy)_y\right)}_{\text{product rule}}$$

$$= 6xy^2 + 2y^{-3} + \overbrace{e^{2x^2y}\cdot 2x^2}^{\text{done in } f_y}\cdot 4xy + e^{2x^2y}\cdot 4x$$

$$= 6xy^2 + 2y^{-3} + e^{2x^2y}(8x^3y + 4x).$$

$$f_{yx} = (f_y)_x = \left(3x^2y^2 + 2xy^{-3} + e^{2x^2y}\cdot 2x^2\right)_x$$

$$= 6xy^2 + 2y^{-3} + \underbrace{\left(\underbrace{(e^{2x^2y})_x}_{\text{done in } f_y}\cdot 2x^2 + e^{2x^2y}\cdot(2x^2)_x\right)}_{\text{product rule}}$$

$$= 6xy^2 + 2y^{-3} + \left(e^{2x^2y}\cdot 4xy\cdot 2x^2 + e^{2x^2y}\cdot 4x\right)$$

$$= 6xy^2 + 2y^{-3} + e^{2x^2y}(8x^3y + 4x)$$

We see that, as promised, $f_{xy} = f_{yx}$.