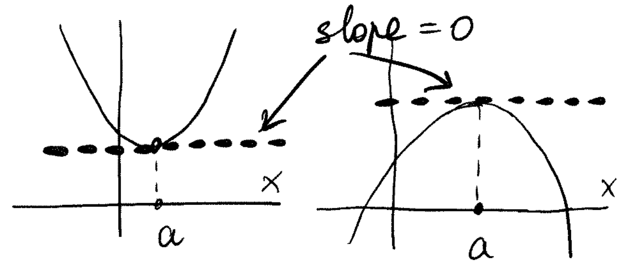
Lecture 15 — Maxima, minima, & saddle points of functions of 2 variables, $f(x,y)$.

Notes:

1) Maxim**um**, minim**um**, extrem**um** (a maximum or minimum) — each of these words is a **singular** form. Maxim**a**, minim**a**, extrem**a** — are **plural** forms.
2) We will consider only **smooth** curves, so that no corners, edges, or cusps will occur, and all 1st- and 2nd-order derivatives will exist.
3) All extrema that we will consider will be local extrema. In contrast, absolute extrema (considered for $y=f(x)$ in Chap. 4) is a more complex topic and will not be considered in this course.

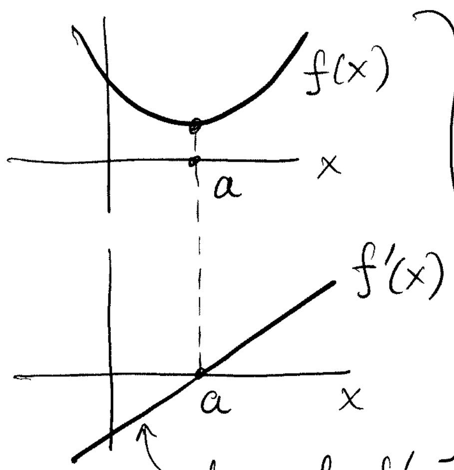
① Review of extrema of $y=f(x)$ (from Chap. 4)

- If $y=f(x)$ has an extremum (i.e., a maximum or a minimum) at $x=a$, then
$$f'(a)=0.$$



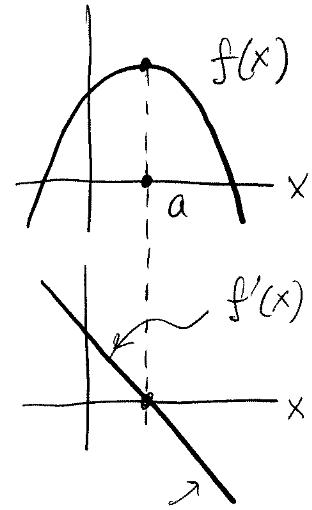
- Given $f'(a)=0$, we **cannot** always tell if we got a maximum, a minimum, or no extremum at all. The so-called 2nd-derivative test can be helpful, but it does not work in 100% cases.

The **2nd-derivative test** gives us an answer when:



slope of f', i.e. $(f')'$, >0

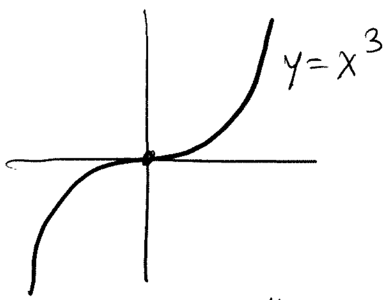
$f'(a)=0$ & $f''(a)>0$ $\Downarrow$ **min**



slope of f', i.e. $(f')'$, <0

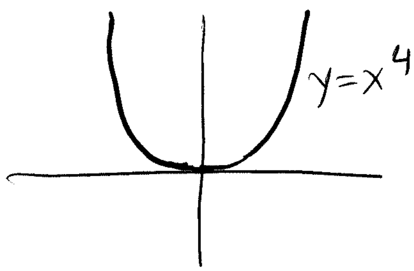
$f'(a)=0$ & $f''(a)<0$ $\Downarrow$ **max**

But if $f'(a)=0$ & $f''(a)=0$, we **cannot tell** whether we have an extremum or not:



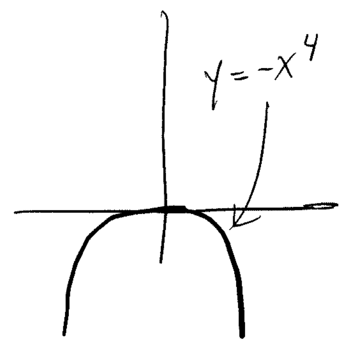
$f'(0)=0$ & $f''(0)=0$

no extremum



$f'(0)=0$ & $f''(0)=0$

minimum



$f'(0)=0$ & $f''(0)=0$

maximum

To summarize the process of finding extrema for $y=f(x)$:

- Find where $f'(a)=0$;
- Check $f''(a)$ >0, <0, or $=0$:

$$\left.\begin{array}{l} f''(a)>0 \Rightarrow \text{min} \\ f''(a)<0 \Rightarrow \text{max} \\ f''(a)=0 \Rightarrow \text{no conclusion} \end{array}\right\} \text{2nd-derivative test.}$$

For $z=f(x,y)$, the process of finding extrema is similar, but the 2nd-derivative test is more complicated.

② Finding extrema of $z = f(x,y)$

— Find point(s) $(x,y) = (a,b)$ where $f_x(a,b) = 0$ & $f_y(a,b) = 0$.

— Run the 2nd-derivative test, which now consists of 2 steps:

<u>Step 1</u>: Compute $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$, $C = f_{yy}(a,b)$

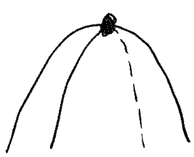
<u>Step 2</u>: Compute $D = AC - B^2$

<u>Case 1</u>: $D > 0$ & A (or C) < 0 $\Rightarrow$ max

<u>Case 2</u>: $D > 0$ & A (or C) > 0 $\Rightarrow$ min

<u>Case 3</u>: $D < 0$ $\Rightarrow$ saddle point (a hybrid type of extremium = max in one direction & min in the other).

<u>Case 4</u>: $D = 0$ $\Rightarrow$ test is inconclusive (cannot tell what extremum there is, if any).



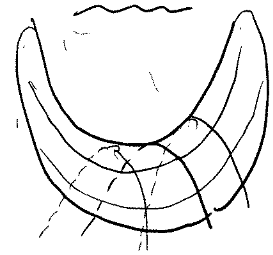
$D > 0$ & A (or C) < 0

max

$D > 0$ & A (or C) > 0

min

Saddle



$D < 0$.

③ Applications

<u>Ex. 1</u> Maximizing profit

In Ex. 1(g) of Lec. 13 & Ex. 2 of Lec. 14 we considered the profit made by a store on selling two competing products. Find the prices that maximize that profit.

Sol'n: 1) From Ex. 2 of Lec. 14 we know:

$$P_p(p,q) = 360 - 6p + 2q, \quad P_q(p,q) = 280 + 2p - 4q,$$

where p, q are the prices of the products and P is the profit from selling both of them.

(Note that if a problem were to be done from scratch, then $P(p,q)$, $P_p(p,q)$ and $P_q(p,q)$ would need to be computed at this step of the solution.)

2) Set $P_p = 0$ & $P_q = 0$:

$$\begin{cases} 360 - 6p + 2q = 0 \\ 280 + 2p - 4q = 0 \end{cases} \xrightarrow[\text{to the other side}]{\text{move constants}} \begin{cases} 6p - 2q = 360 \\ -2p + 4q = 280 \end{cases} \xrightarrow[\text{by 2}]{\text{Divide out}} \begin{cases} 3p - q = 180 \\ -p + 2q = 140 \end{cases}$$

Solve for q from Eq. (1) and substitute into Eq. (2):

$q = 3p - 180$,

$-p + 2(3p - 180) = 140 \Rightarrow 5p - 360 = 140 \Rightarrow 5p = 500 \Rightarrow p = 100.$

$q = 3 \cdot 100 - 180 = 120$

Thus, **an extremum can be @ $p = 100$, $q = 120$.**

3) To check that we indeed have a maximum there, **use the 2nd-derivative test**:

Step 1: $A = P_{pp} = (360 - 6p + 2q)_p = -6$

$B = P_{pq} = (360 - 6p + 2q)_q = 2$

$C = P_{qq} = (280 + 2p - 4q)_q = -4$

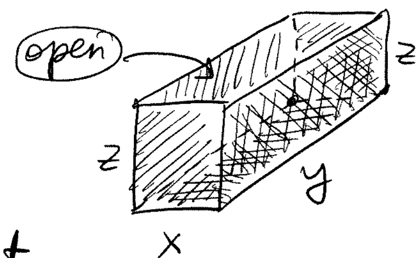
Note: In this problem we did not need to substitute $p = 100$, $p = 120$ into $P_{pp}(p,q)$ etc. since all 2nd partial derivatives = const.

Step 2 $\begin{cases} D = AC - B^2 = (-6)(-4) - 2^2 = 20 > 0, \\ A = -6 < 0, \end{cases}$

$\to$ maximum.

Ex. 2 In Ex. 1(c) of Lec. 13 we found the amount of material needed to build an open-top box:

$$\underset{\substack{\uparrow \\ \text{surface} \\ \text{area}}}{S} = xy + 2xz + 2yz$$



Find the **dimensions of the box that** **minimize the amount of material needed**, given that the box must have volume $V = 108$ cubic units.

Sol'n: 1) Use $V = xyz$ and $V = 108$ to solve for one of the variables, say, z: $z = V/(xy) = 108/(xy)$.

2) Substitute into the expression for S:

$$S = xy + 2\cancel{x} \cdot \frac{V}{\cancel{x}y} + 2\cancel{y} \cdot \frac{V}{x\cancel{y}} = xy + \frac{216}{y} + \frac{216}{x}$$

3) Find S_x, S_y:

$$S_x = y + 0 - \frac{216}{x^2}, \quad S_y = x - \frac{216}{y^2} + 0.$$

4) Set them to 0 to find possible extrema:

$$S_x = 0 \Rightarrow y - 216/x^2 = 0 \qquad \text{(Eq. 1)}$$

$$S_y = 0 \Rightarrow x - 216/y^2 = 0 \qquad \text{(Eq. 2)}$$

Solve for one variable; say, from (Eq. 1): $\boxed{y = 216/x^2}$.

Substitute into (Eq. 2): $x - \dfrac{216}{(216/x^2)^2} = 0 \Rightarrow$

$$\Rightarrow x - \frac{\cancel{216}\cdot x^4}{216^{\cancel{2}}} = 0 \Rightarrow x = \frac{x^4}{216}.$$

Since $x \neq 0$, we can cancel by it $\Rightarrow 216 = x^3 \Rightarrow \underline{\underline{x=6}}$.

Now substitute this into the equation we got for y:

$\underline{\underline{y}} = 216/6^2 = \underline{\underline{6}}$.

<u>Note</u>: Even though "after the fact" we can look back at the picture of the box and see why $x=y$ makes sense, we <u>are not allowed</u> to assume that from the beginning.

5) Verify that these values give a <u>minimum</u> of S.

Use the 2nd-derivative test.

<u>Step 1</u>: Compute $S_{xx} = (S_x)_x = \left(y - \frac{216}{x^2}\right)_x = (y - 216\cdot x^{-2})_x$

$= 0 - 216\cdot(-2)\cdot x^{-3} = 432/x^3.$

$A = S_{xx}(\underset{x}{6}, \underset{y}{6}) = 432/6^3 = 2 \Rightarrow A=2$

$S_{xy} = (S_x)_y = (y - 216\cdot x^{-3})_y = 1 - 0 = 1 \Rightarrow B=1$

(no need to sub. in $x=6, y=6$ since $S_{xy} = 1 = \text{const}$).

$S_{yy} = (x - 216\cdot y^{-2})_y = -216\cdot(-2)y^{-3} \Rightarrow$

$C = S_{yy}(6,6) = 432/6^3 = 2 \Rightarrow C=2$

<u>Step 2</u>: $\left.\begin{array}{l} D = AC - B^2 = 2\cdot 2 - 1^2 = 3 > 0 \\ A = 2 > 0 \end{array}\right\} \Rightarrow$

point $(x=6, y=6)$ indeed gives the <u>minimum</u> for the amount of material.

6) If we need to find z, we recall that $z = \frac{V}{xy}$

$= 108/(6\cdot 6) = 3.$