

may happen. In Problems 20–22 determine the behavior of all solutions of the given differential equation as  $t \rightarrow 0$ , and in Problem 23 determine the behavior of all solutions as  $t \rightarrow \pi/2$ .

$$20. \frac{dy}{dt} + \frac{1}{t}y = \frac{1}{t^2}$$

$$21. \frac{dy}{dt} + \frac{1}{\sqrt{t}}y = e^{\sqrt{t}/2}$$

$$22. \frac{dy}{dt} + \frac{1}{t}y = \cos t + \frac{\sin t}{t}$$

$$23. \frac{dy}{dt} + y \tan t = \sin t \cos t.$$

### 1.3 The Van Meegeren art forgeries

After the liberation of Belgium in World War II, the Dutch Field Security began its hunt for Nazi collaborators. They discovered, in the records of a firm which had sold numerous works of art to the Germans, the name of a banker who had acted as an intermediary in the sale to Goering of the painting “Woman Taken in Adultery” by the famed 17th century Dutch painter Jan Vermeer. The banker in turn revealed that he was acting on behalf of a third rate Dutch painter H. A. Van Meegeren, and on May 29, 1945 Van Meegeren was arrested on the charge of collaborating with the enemy. On July 12, 1945 Van Meegeren startled the world by announcing from his prison cell that he had never sold “Woman Taken in Adultery” to Goering. Moreover, he stated that this painting and the very famous and beautiful “Disciples at Emmaus”, as well as four other presumed Vermeers and two de Hooghs (a 17th century Dutch painter) were his own work. Many people, however, thought that Van Meegeren was only lying to save himself from the charge of treason. To prove his point, Van Meegeren began, while in prison, to forge the Vermeer painting “Jesus Amongst the Doctors” to demonstrate to the skeptics just how good a forger of Vermeer he was. The work was nearly completed when Van Meegeren learned that a charge of forgery had been substituted for that of collaboration. He, therefore, refused to finish and age the painting so that hopefully investigators would not uncover his secret of aging his forgeries. To settle the question an international panel of distinguished chemists, physicists and art historians was appointed to investigate the matter. The panel took x-rays of the paintings to determine whether other paintings were underneath them. In addition, they analyzed the pigments (coloring materials) used in the paint, and examined the paintings for certain signs of old age.

Now, Van Meegeren was well aware of these methods. To avoid detection, he scraped the paint from old paintings that were not worth much, just to get the canvas, and he tried to use pigments that Vermeer would have used. Van Meegeren also knew that old paint was extremely hard, and impossible to dissolve. Therefore, he very cleverly mixed a chemical, phenoformaldehyde, into the paint, and this hardened into bakelite when the finished painting was heated in an oven.

However, Van Meegeren was careless with several of his forgeries, and the panel of experts found traces of the modern pigment cobalt blue. In addition, they also detected the phenoformaldehyde, which was not discovered until the turn of the 19th century, in several of the paintings. On the basis of this evidence Van Meegeren was convicted, of forgery, on October 12, 1947 and sentenced to one year in prison. While in prison he suffered a heart attack and died on December 30, 1947.

However, even following the evidence gathered by the panel of experts, many people still refused to believe that the famed “Disciples at Emmaus” was forged by Van Meegeren. Their contention was based on the fact that the other alleged forgeries and Van Meegeren’s nearly completed “Jesus Amongst the Doctors” were of a very inferior quality. Surely, they said, the creator of the beautiful “Disciples at Emmaus” could not produce such inferior pictures. Indeed, the “Disciples at Emmaus” was certified as an authentic Vermeer by the noted art historian A. Bredius and was bought by the Rembrandt Society for \$170,000. The answer of the panel to these skeptics was that because Van Meegeren was keenly disappointed by his lack of status in the art world, he worked on the “Disciples at Emmaus” with the fierce determination of proving that he was better than a third rate painter. After producing such a masterpiece his determination was gone. Moreover, after seeing how easy it was to dispose of the “Disciples at Emmaus” he devoted less effort to his subsequent forgeries. This explanation failed to satisfy the skeptics. They demanded a thoroughly scientific and conclusive proof that the “Disciples at Emmaus” was indeed a forgery. This was done recently in 1967 by scientists at Carnegie Mellon University, and we would now like to describe their work.

The key to the dating of paintings and other materials such as rocks and fossils lies in the phenomenon of radioactivity discovered at the turn of the century. The physicist Rutherford and his colleagues showed that the atoms of certain “radioactive” elements are unstable and that within a given time period a fixed proportion of the atoms spontaneously disintegrate to form atoms of a new element. Because radioactivity is a property of the atom, Rutherford showed that the radioactivity of a substance is directly proportional to the number of atoms of the substance present. Thus, if  $N(t)$  denotes the number of atoms present at time  $t$ , then  $dN/dt$ , the number of atoms that disintegrate per unit time is proportional to  $N$ , that is,

$$\frac{dN}{dt} = -\lambda N. \quad (1)$$

The constant  $\lambda$  which is positive, is known as the decay constant of the substance. The larger  $\lambda$  is, of course, the faster the substance decays. One measure of the rate of disintegration of a substance is its *half-life* which is defined as the time required for half of a given quantity of radioactive atoms to decay. To compute the half-life of a substance in terms of  $\lambda$ , assume that at time  $t_0$ ,  $N(t_0) = N_0$ . Then, the solution of the initial-value

problem  $dN/dt = -\lambda N$ ,  $N(t_0) = N_0$  is

$$N(t) = N_0 \exp\left(-\lambda \int_{t_0}^t ds\right) = N_0 e^{-\lambda(t-t_0)}$$

or  $N/N_0 = \exp(-\lambda(t-t_0))$ . Taking logarithms of both sides we obtain that

$$-\lambda(t-t_0) = \ln \frac{N}{N_0}. \quad (2)$$

Now, if  $N/N_0 = \frac{1}{2}$  then  $-\lambda(t-t_0) = \ln \frac{1}{2}$  so that

$$(t-t_0) = \frac{\ln 2}{\lambda} = \frac{0.6931}{\lambda}. \quad (3)$$

Thus, the half-life of a substance is  $\ln 2$  divided by the decay constant  $\lambda$ . The dimension of  $\lambda$ , which we suppress for simplicity of writing, is reciprocal time. If  $t$  is measured in years then  $\lambda$  has the dimension of reciprocal years, and if  $t$  is measured in minutes, then  $\lambda$  has the dimension of reciprocal minutes. The half-lives of many substances have been determined and recorded. For example, the half-life of carbon-14 is 5568 years and the half-life of uranium-238 is 4.5 billion years.

Now the basis of “radioactive dating” is essentially the following. From Equation (2) we can solve for  $t-t_0 = 1/\lambda \ln(N_0/N)$ . If  $t_0$  is the time the substance was initially formed or manufactured, then the age of the substance is  $1/\lambda \ln(N_0/N)$ . The decay constant  $\lambda$  is known or can be computed, in most instances. Moreover, we can usually evaluate  $N$  quite easily. Thus, if we knew  $N_0$  we could determine the age of the substance. But this is the real difficulty of course, since we usually do not know  $N_0$ . In some instances though, we can either determine  $N_0$  indirectly, or else determine certain suitable ranges for  $N_0$ , and such is the case for the forgeries of Van Meegeren.

We begin with the following well-known facts of elementary chemistry. Almost all rocks in the earth’s crust contain a small quantity of uranium. The uranium in the rock decays to another radioactive element, and that one decays to another and another, and so forth (see Figure 1) in a series of elements that results in lead, which is not radioactive. The uranium (whose half-life is over four billion years) keeps feeding the elements following it in the series, so that as fast as they decay, they are replaced by the elements before them.

Now, all paintings contain a small amount of the radioactive element lead-210 ( $^{210}\text{Pb}$ ), and an even smaller amount of radium-226 ( $^{226}\text{Ra}$ ), since these elements are contained in white lead (lead oxide), which is a pigment that artists have used for over 2000 years. For the analysis which follows, it is important to note that white lead is made from lead metal, which, in turn, is extracted from a rock called lead ore, in a process called smelting. In this process, the lead-210 in the ore goes along with the lead metal. However, 90–95% of the radium and its descendants are removed with

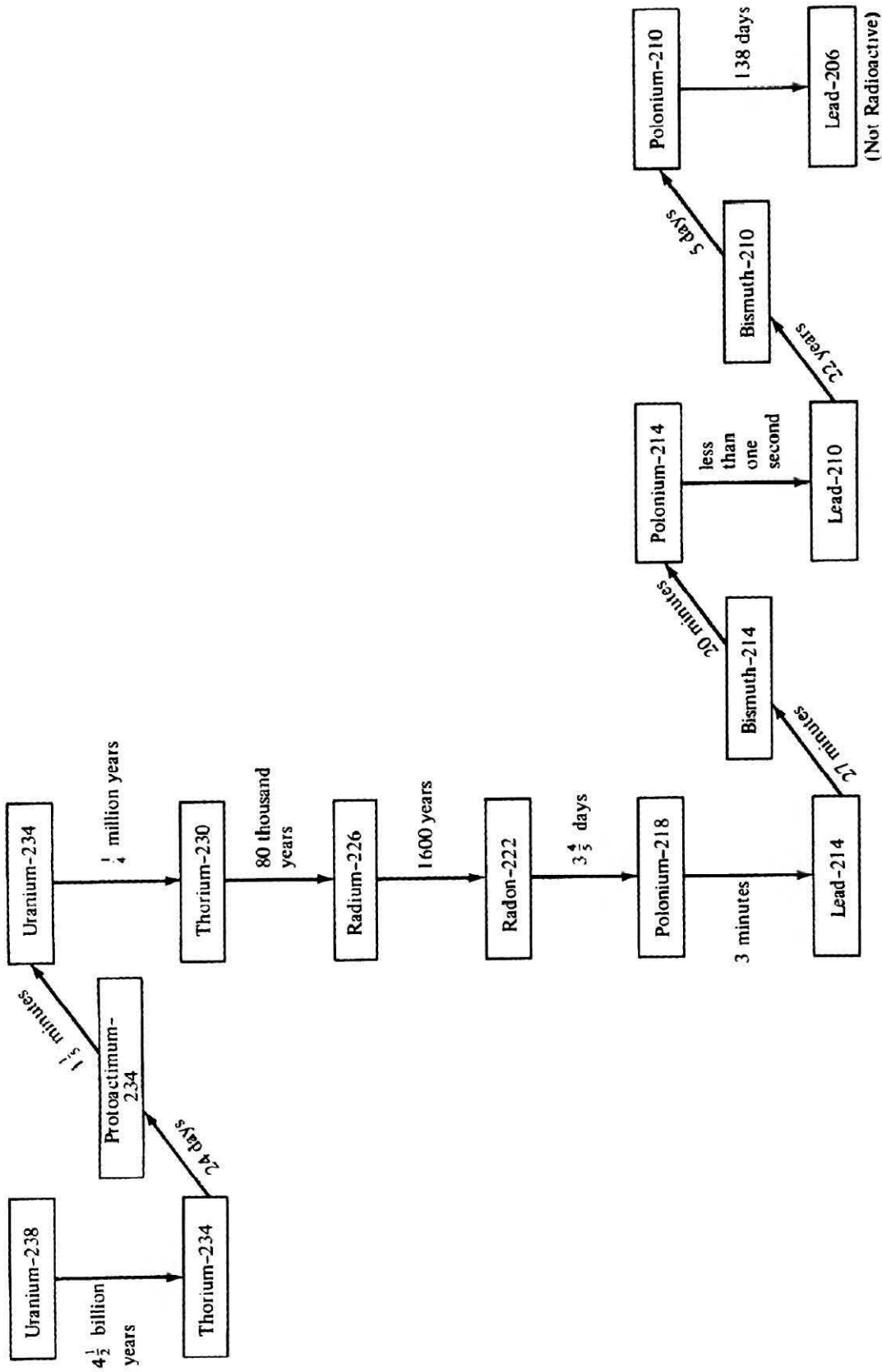


Figure 1. The Uranium series. (The times shown on the arrows are the half-lives of each step.)

other waste products in a material called slag. Thus, most of the supply of lead-210 is cut off and it begins to decay very rapidly, with a half-life of 22 years. This process continues until the lead-210 in the white lead is once more in radioactive equilibrium with the small amount of radium present, i.e. the disintegration of the lead-210 is exactly balanced by the disintegration of the radium.

Let us now use this information to compute the amount of lead-210 present in a sample in terms of the amount originally present at the time of manufacture. Let  $y(t)$  be the amount of lead-210 per gram of white lead at time  $t$ ,  $y_0$  the amount of lead-210 per gram of white lead present at the time of manufacture  $t_0$ , and  $r(t)$  the number of disintegrations of radium-226 per minute per gram of white lead, at time  $t$ . If  $\lambda$  is the decay constant for lead-210, then

$$\frac{dy}{dt} = -\lambda y + r(t), \quad y(t_0) = y_0. \quad (4)$$

Since we are only interested in a time period of at most 300 years we may assume that the radium-226, whose half-life is 1600 years, remains constant, so that  $r(t)$  is a constant  $r$ . Multiplying both sides of the differential equation by the integrating factor  $\mu(t) = e^{\lambda t}$  we obtain that

$$\frac{d}{dt} e^{\lambda t} y = r e^{\lambda t}.$$

Hence

$$e^{\lambda t} y(t) - e^{\lambda t_0} y_0 = \frac{r}{\lambda} (e^{\lambda t} - e^{\lambda t_0})$$

or

$$y(t) = \frac{r}{\lambda} (1 - e^{-\lambda(t-t_0)}) + y_0 e^{-\lambda(t-t_0)}. \quad (5)$$

Now  $y(t)$  and  $r$  can be easily measured. Thus, if we knew  $y_0$  we could use Equation (5) to compute  $(t - t_0)$  and consequently, we could determine the age of the painting. As we pointed out, though, we cannot measure  $y_0$  directly. One possible way out of this difficulty is to use the fact that the original quantity of lead-210 was in radioactive equilibrium with the larger amount of radium-226 in the ore from which the metal was extracted. Let us, therefore, take samples of different ores and count the number of disintegrations of the radium-226 in the ores. This was done for a variety of ores and the results are given in Table 1 below. These numbers vary from 0.18 to 140. Consequently, the number of disintegrations of the lead-210 per minute per gram of white lead at the time of manufacture will vary from 0.18 to 140. This implies that  $y_0$  will also vary over a very large interval, since the number of disintegrations of lead-210 is proportional to the amount present. Thus, we cannot use Equation (5) to obtain an accurate, or even a crude estimate, of the age of a painting.

Table 1. Ore and ore concentrate samples. All disintegration rates are per gram of white lead.

Description and Source	Disintegrations per minute of $^{226}\text{Ra}$
Ore concentrate (Oklahoma-Kansas)	4.5
Crushed raw ore (S.E. Missouri)	2.4
Ore concentrate (S.E. Missouri)	0.7
Ore concentrate (Idaho)	2.2
Ore concentrate (Idaho)	0.18
Ore concentrate (Washington)	140.0
Ore concentrate (British Columbia)	1.9
Ore concentrate (British Columbia)	0.4
Ore concentrate (Bolivia)	1.6
Ore concentrate (Australia)	1.1

However, we can still use Equation (5) to distinguish between a 17th century painting and a modern forgery. The basis for this statement is the simple observation that if the paint is very old compared to the 22 year half-life of lead, then the amount of radioactivity from the lead-210 in the paint will be nearly equal to the amount of radioactivity from the radium in the paint. On the other hand, if the painting is modern (approximately 20 years old, or so) then the amount of radioactivity from the lead-210 will be much greater than the amount of radioactivity from the radium.

We make this argument precise in the following manner. Let us assume that the painting in question is either very new or about 300 years old. Set  $t - t_0 = 300$  years in (5). Then, after some simple algebra, we see that

$$\lambda y_0 = \lambda y(t)e^{300\lambda} - r(e^{300\lambda} - 1). \quad (6)$$

If the painting is indeed a modern forgery, then  $\lambda y_0$  will be absurdly large. To determine what is an absurdly high disintegration rate we observe (see Exercise 1) that if the lead-210 decayed originally (at the time of manufacture) at the rate of 100 disintegrations per minute per gram of white lead, then the ore from which it was extracted had a uranium content of approximately 0.014 per cent. This is a fairly high concentration of uranium since the average amount of uranium in rocks of the earth's crust is about 2.7 parts per million. On the other hand, there are some very rare ores in the Western Hemisphere whose uranium content is 2-3 per cent. To be on the safe side, we will say that a disintegration rate of lead-210 is certainly absurd if it exceeds 30,000 disintegrations per minute per gram of white lead.

To evaluate  $\lambda y_0$ , we must evaluate the present disintegration rate,  $\lambda y(t)$ , of the lead-210, the disintegration rate  $r$  of the radium-226, and  $e^{300\lambda}$ . Since the disintegration rate of polonium-210 ( $^{210}\text{Po}$ ) equals that of lead-210 after several years, and since it is easier to measure the disintegration rate of polonium-210, we substitute these values for those of lead-210. To compute

$e^{300\lambda}$ , we observe from (3) that  $\lambda = (\ln 2/22)$ . Hence

$$e^{300\lambda} = e^{(300/22)\ln 2} = 2^{(150/11)}.$$

The disintegration rates of polonium-210 and radium-226 were measured for the “Disciples at Emmaus” and various other alleged forgeries and are given in Table 2 below.

Table 2. Paintings of questioned authorship. All disintegration rates are per minute, per gram of white lead.

Description	$^{210}\text{Po}$ disintegration	$^{226}\text{Ra}$ disintegration
“Disciples at Emmaus”	8.5	0.8
“Washing of Feet”	12.6	0.26
“Woman Reading Music”	10.3	0.3
“Woman Playing Mandolin”	8.2	0.17
“Lace Maker”	1.5	1.4
“Laughing Girl”	5.2	6.0

If we now evaluate  $\lambda y_0$  from (6) for the white lead in the painting “Disciples at Emmaus” we obtain that

$$\begin{aligned}\lambda y_0 &= (8.5)2^{150/11} - 0.8(2^{150/11} - 1) \\ &= 98,050\end{aligned}$$

which is unacceptably large. Thus, this painting must be a modern forgery. By a similar analysis, (see Exercises 2–4) the paintings “Washing of Feet”, “Woman Reading Music” and “Woman Playing Mandolin” were indisputably shown to be faked Vermeers. On the other hand, the paintings “Lace Maker” and “Laughing Girl” cannot be recently forged Vermeers, as claimed by some experts, since for these two paintings, the polonium-210 is very nearly in radioactive equilibrium with the radium-226, and no such equilibrium has been observed in any samples from 19th or 20th century paintings.

### References

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