A more realistic model of one-dimensional motion is one that incorporates the effects of air resistance. As an object moves through air, a retarding aerodynamic force is created by the combination of pressure and frictional forces. The air close to the object exerts a normal pressure force upon it. Likewise, friction creates a tangential force that opposes the motion of air past the object. The combination of these effects creates a drag force that acts to reduce the speed of the moving object.

The drag force depends on the velocity $v(t)$ of the object and acts on it in such a way as to reduce its speed, $|v(t)|$. We consider two idealized models of drag force that are consistent with these ideas.

Case 1: Drag Force Proportional to Velocity

Assume that velocity $v(t)$ is positive in the upward direction and that the drag force is proportional to velocity. If k is the positive constant of proportionality, Newton's second law of motion leads to

$$m\frac{dv}{dt} = -mg - kv. \tag{1}$$

Does the model of drag that we have postulated act as we want it to? If the object is moving upward [that is, if $v(t) > 0$], then the drag force $-kv(t)$ is negative and thus acts downward. Conversely, if $v(t) < 0$, then the object is moving downward. In this case, the drag force $-kv(t) = k|v(t)|$ is a positive (upward) force, as it should be. Therefore, whether the object is moving upward or downward, drag acts to slow the object; this drag model is consistent with our ideas of how drag should act. See Figure 2.15.

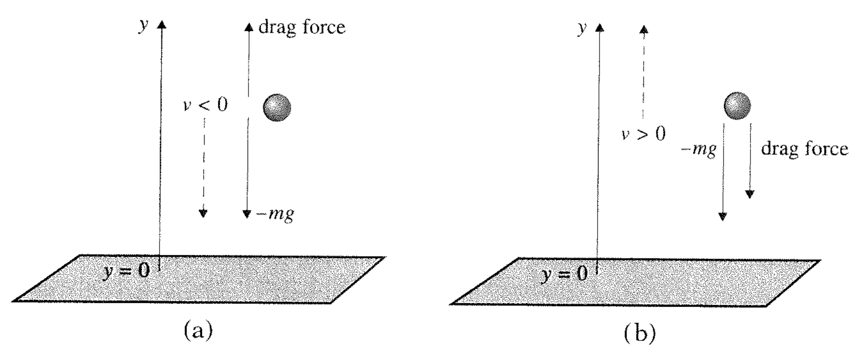

FIGURE 2.15

Assume a drag force of the form $kv(t)$, $k > 0$; see equation (1). (a) When the object is moving downward, drag acts upward and tends to slow the object. (b) When the object is moving upward, drag acts downward and likewise tends to slow the object.

This drag model leads to equation (1), a first order linear constant coefficient equation; we can solve it using the ideas of Section 2.2. Before we do, however, let's consider the question of equilibrium solutions to see if the model makes sense in that regard. The only constant solution of equation (1) is $v(t) = -mg/k$. At the velocity $v(t) = -mg/k$, the drag force and gravitational force acting on the object are equal and opposite. This equilibrium velocity is often referred to as the **terminal velocity** of the object.

The initial value problem corresponding to equation (1) is

$$m\frac{dv}{dt} = -mg - kv, \qquad v(0) = v_0. \tag{2}$$

Solving initial value problem (2), we obtain

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-(k/m)t}.$$

From this explicit solution it is clear that, for any initial velocity v_0, $v(t)$ tends to the terminal velocity

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k}.$$

The concept of terminal velocity is a mathematical abstraction since, in any application, the time interval of interest is finite. However, whether the object is initially moving up or down, it eventually begins to fall and its velocity approaches terminal velocity as the time of falling increases. The velocity that the object actually has the instant before it strikes the ground is called the **impact velocity**. Once impact occurs, the mathematical model (2) is no longer applicable.

Case 2: Drag Force Proportional to the Square of Velocity

A drag force having magnitude $\kappa v^2(t)$ (where the constant of proportionality κ is assumed to be positive) has been found in many cases to be a reasonably good approximation to reality over a range of velocities. However, since it involves an even power of velocity, incorporating this model of drag into the equations of motion requires more care.

Drag must act to reduce the speed of the moving object. Therefore, if the object is moving upward, the drag force acts downward and should be $-\kappa v^2(t)$. If the object is moving downward, the drag force acts upward and should be $\kappa v^2(t)$. In other words, when we use this model for drag, Newton's second law leads to

$$m\frac{dv}{dt} = -mg - \kappa v^2, \qquad v(t) \ge 0,$$

$$m\frac{dv}{dt} = -mg + \kappa v^2, \qquad v(t) \le 0.$$

Here again we can ask about equilibrium solutions and see if the answer makes sense physically. Note that no equilibrium solution exists if $v(t) > 0$ since $-mg - \kappa v^2$ is never zero. When $v(t) < 0$, however, there is an equilibrium solution:

$$v(t) = -\sqrt{\frac{mg}{\kappa}}.$$

This equilibrium solution is again a terminal velocity corresponding to downward motion; at this velocity, drag and gravity exert equal and opposite forces.

Each of the preceding equations is a first order separable equation. If the problem involves a falling (rising) object, then velocity is always nonpositive (nonnegative) and a single equation is valid for the entire problem (that is, over the entire t-interval of interest). If, however, the problem involves both upward and downward motion, then both equations will ultimately be needed. The first equation must be used to model the upward dynamics, the behavior

Imposing the initial condition, we find the implicit solution

$$\frac{v^2}{2} = GM_e\left(\frac{1}{r} - \frac{1}{R_e + h}\right).$$

Since separation distance is decreasing with time, velocity is negative. Therefore, an explicit solution of the problem is

$$v = -\sqrt{2GM_e\left(\frac{1}{r} - \frac{1}{R_e + h}\right)}.$$

The impact velocity is found by evaluating velocity at $r = R_e$:

$$v_{\text{impact}} = -\sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{R_e + h}\right)}.$$

Using the values given, we obtain $v_{\text{impact}} = -1952$ m/s (a speed of about 4350 mph). [Note: The impact velocity does not depend on the mass of the object, but only on its distance from Earth when it is released.] ❖

In the Exercises, we ask you to consider the same problem in the presence of a drag force that is proportional to the square of the velocity. In particular, the governing differential equation becomes

$$m\frac{dv}{dt} = -\frac{GmM_e}{r^2} + \kappa v^2. \tag{13}$$

After the independent variable is changed from time to distance, the resulting differential equation can be recast as a Bernoulli equation and then solved; see Exercise 16.

EXERCISES

1. An object of mass m is dropped from a high altitude. How long will it take the object to achieve a velocity equal to one half of its terminal velocity if the drag force is assumed to be proportional to the velocity?
2. A drag chute must be designed to reduce the speed of a 3000-lb dragster from 220 mph to 50 mph in 4 sec. Assume that the drag force is proportional to the velocity.
   (a) What value of the drag coefficient k is needed to accomplish this?
   (b) How far will the dragster travel in the 4-sec interval?
3. Repeat Exercise 2 for the case in which the drag force is proportional to the square of the velocity. Determine both the drag coefficient κ and the distance traveled.
4. A projectile of mass m is launched vertically upward from ground level at time $t = 0$ with initial velocity v_0 and is acted upon by gravity and air resistance. Assume the drag force is proportional to velocity, with drag coefficient k. Derive an expression for the time, t_m, when the projectile achieves its maximum height.
5. Derive an expression for the maximum height, $y_m = y(t_m)$, achieved in Exercise 4.
6. An object of mass m is dropped from a high altitude and is subjected to a drag force proportional to the square of its velocity. How far must the object fall before its velocity reaches one half its terminal velocity?
7. An object is dropped from altitude y_0. Determine the impact velocity if air resistance is neglected—that is, if we assume no drag force.

16. The motion of a body of mass m, gravitationally attracted to Earth in the presence of a resisting drag force proportional to the square of its velocity, is given by

$$m\frac{dv}{dt} = -\frac{GmM_e}{r^2} + \kappa v^2$$

[recall equation (13)]. In this equation, r is the radial distance of the body from the center of Earth, G is the universal gravitational constant, M_e is the mass of Earth, and $v = dr/dt$. Note that the drag force is positive, since it acts in the positive r direction.

(a) Assume that the body is released from rest at an altitude h above the surface of Earth. Recast the differential equation so that distance r is the independent variable. State an appropriate initial condition for the new problem.

(b) Show that the impact velocity can be expressed as

$$v_{\text{impact}} = -\left[2GM_e \int_0^h \frac{e^{-2(\kappa/m)s}}{(R_e+s)^2}\,ds\right]^{1/2},$$

where R_e represents the radius of Earth. (The minus sign reflects the fact that $v = dr/dt < 0$.)

17. On August 24, 1894, Pop Shriver of the Chicago White Stockings caught a baseball dropped (by Clark Griffith) from the top of the Washington Monument. The Washington Monument is 555 ft tall and a baseball weighs $5\frac{1}{8}$ oz.

(a) If we ignore air resistance and assume the baseball was acted upon only by gravity, how fast would the baseball have been traveling when it was 7 ft above the ground?

(b) Suppose we now include air resistance in our model, assuming that the drag force is proportional to velocity with a drag coefficient $k = 0.0018$ lb-sec/ft. How fast is the baseball traveling in this case when it is 7 ft above the ground?

18. A 180-lb skydiver drops from a hot-air balloon. After 10 sec of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 sec, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-lb person will reach a terminal velocity of -10 mph.

(a) What is the speed of the skydiver immediately before the parachute is opened?

(b) What is the parachutist's impact velocity?

(c) At what altitude was the parachute opened?

(d) What is the balloon's altitude?

19. When modeling the action of drag chutes and parachutes, we have assumed that the chute opens instantaneously. Real devices take a short amount of time to fully open and deploy.

In this exercise, we try to assess the importance of this distinction. Consider again the assumptions of Exercise 2. A 3000-lb dragster is moving on a straight track at a speed of 220 mph when, at time $t = 0$, the drag chute is opened. If we assume that the drag force is proportional to velocity and that the chute opens instantaneously, the differential equation to solve is $mv' = -kv$.

If we assume a short deployment time to open the chute, a reasonable differential equation might be $mv' = -k(\tanh t)v$. Since $\tanh(0) = 0$ and $\tanh(1) \approx 0.76$, it will take about 1 sec for the chute to become 76% deployed in this model.

Assume $k = 25$ lb-sec/ft. Solve the two differential equations and determine in each case how long it takes the vehicle to slow to 50 mph. Which time do you anticipate will be larger? (Explain.) Is the idealization of instantaneous chute deployment realistic?