

MATH 230

Elementary Differential Equations

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Lecture 1. Introduction

① Motivation

Ex. 1 a) An object starts at y_0 and moves along y -axis with speed v .

Find location $y(t)$ at any t .



Sol'n: $y(t) = y_0 + vt$.

b) Q: What if $v \neq \text{const}$? (1)

We know: $y'(t) = v(t)$ ← integrate

$$y(t) = \int v(t) dt + C \quad (2)$$

↑ explicitly account for arb. constant.

Constant C is found from the initial condition.

However, we now use an integration method which will find C right away. (we'll use the same method many times in this course.)

1-2

$$\int_0^t y'(t_1) dt_1 = \int_0^t v(t_1) dt_1$$

we will explain this notation later

By Fund. Thm. of Calculus = $y(t) - y(0)$.
so:

$$y(t) - y(0) = \int_0^t v(t_1) dt_1,$$

$$y(t) = y_0 + \int_0^t v(t_1) dt_1,$$

(3)

Moral: Our goal is:

Given the diff. eqn. $\frac{dy}{dt} = f(t, y)$ (4a)

and initial condition $y(t_0) = y_0$ (4b)

We want to find solution $y(t)$.

Note 1 The combination of diff. eqn., (4a), and initial condition, (4b), is called initial value problem (IVP). (DE)

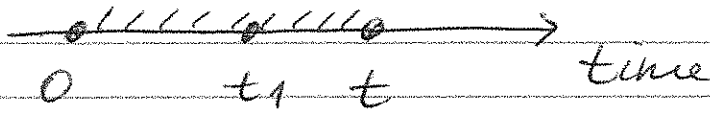
Note 2 When we solve any DE, we get an arbitrary constant, C (see (2)). Its value is found from the initial cond.; e.g., in (3), $C = y_0$.

Note 3 (about notations)

When we integrate $v(t)$ from 0 to t , we write:

$$\int_0^t v(t_1) dt_1$$

this is the time up to which we integrate



This is a "dummy" variable $(0 \leq t_1 \leq t)$.

It must have a different name

than the finite integration time.

Sometimes, we may use \tilde{t} (t -tilde) instead of t_1 :

$$\int_0^t v(\tilde{t}) d\tilde{t}$$

Note 4 DE (4a) is more complicated than DE (1). Indeed, if we write its solution as:

$$y(t) = \int f(t, y(t)) dt + C, \quad (5)$$

we see that we cannot integrate because we do not know the unknown solution $y(t)$ inside $f(t, y)$.

So we need to develop methods of how to find $y(t)$ other than by (5).

② Examples of DEs

Ex. 2 Given IVP: $y' = ay$ $y(0) = 3$ (6)

↖ const

a) Show that the solution of this DE is $y = Ce^{at}$ ($C = \text{const}$)

b) Find the solution of the IVP.

Sol'n: a) We are asked to verify a given solution, so we simply compare the lhs and rhs,

lhs: $y' = (Ce^{at})' = C(e^{at})' = C \cdot (at)' \cdot e^{at} = C \cdot a \cdot e^{at}$

↖ Chain Rule

rhs: $a \cdot y = a \cdot Ce^{at}$

lhs = rhs. ✓

b) $Ce^{at} \Big|_{t=0} = 3 \Rightarrow C \cdot e^0 = 3 \Rightarrow C = 3$

$y(t) = 3e^{at}$

Note: DE (6) is the most basic DE which arises in many applications in all branches of science (physics, chemistry, ...)

It basically says:

$$\underbrace{y'}_{\substack{\uparrow \\ \text{rate of change} \\ \text{of } y}} = \underbrace{a}_{\substack{\uparrow \\ \text{is proportional} \\ \text{to}}} \cdot \underbrace{y}_{\substack{\leftarrow \\ \text{instantaneous} \\ \text{value of } y.}}$$

For example:

sub-Ex. 2(a)

$$\frac{dm}{dt} = -a \cdot m$$

mass of radioactive material
 (radioactive decay law:
 a given percentage of material decays in
 a unit of time).

sub-Ex. 2(b) ← Newton's Cooling Law

$$\frac{d(T - T_r)}{dt} = -k \cdot (T - T_r)$$


T ↑
 temperature of coffee
 coffee
 $T_r =$ room temp.

temp. of coffee
 in excess of
 room temperature

sub-Ex. 2(c)

$$\frac{dn}{dt} = b \cdot n$$

number of bacteria
 multiplying in presence of unlimited
 supply of food.



③ Some terminology

1) $y'(t) = f(t, y(t))$

$t \leftarrow$ independent variable

$y \leftarrow$ dependent variable

2) Ordinary DE : $\frac{dy}{dt}$ is ordinary derivative.

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Partial DE : $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$
involves partial derivatives of dependent variable $u(x,t)$ w.r.t. two independent variables x, t .

3) Order of DE = order of highest derivative

Ex. 3(a)

$y'' = 1$
 \uparrow 2nd der., \Rightarrow order = 2

Ex. 3(b)

$\sin(y') + \cos(y''') = 0$
 \uparrow 3rd der., \Rightarrow order = 3

Ex. 3(c)

$(y'')^5 + (y')^2 + \ln y = 0$
 \uparrow 2nd der., \Rightarrow order = 2.

4) Autonomous & non-autonomous DE.

$y' = f(y)$ ← autonomous.

$y' = y^2 - \sin(e^y)$.

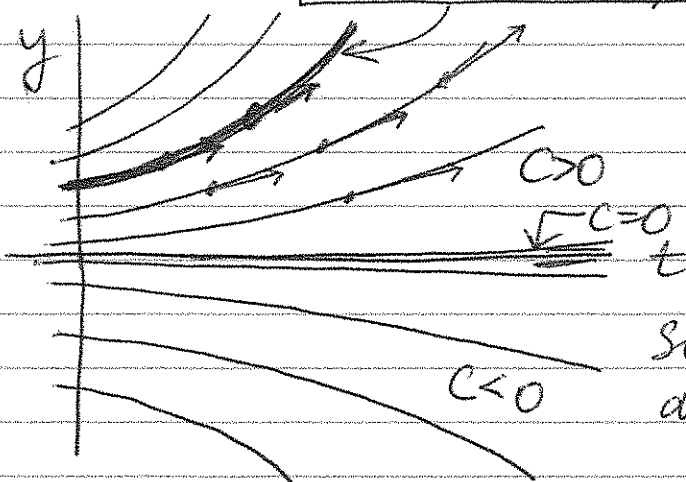
$y' = f(t, y)$ ← non-autonomous

$y' = y^2 - t$

④ Direction fields

Idea: Recall Ex. 2.

IVP has 1 solution $y(t)$.



But DE has ∞ many solutions because of arb. const C.

So sol'ns of DE are many curves.

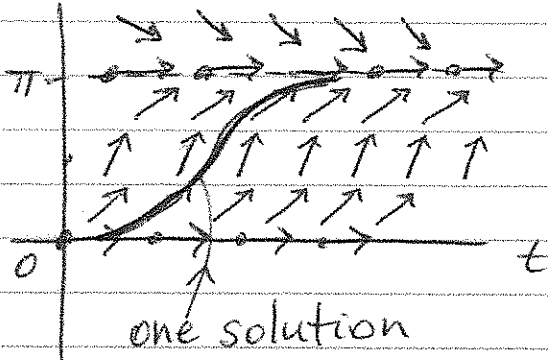
Direction field is the collection of all these curves with arrows at each point indicating the direction of motion.

See also a more detailed version of this example posted next to this Lecture.

1-8

Ex. 4 Sketch the direction field of

$$y' = \sin y$$



Sol'n:

This is an autonomous DE, \Rightarrow

y' is the same along each line $y = \text{const.}$

$$y = 0 \Rightarrow \sin y = 0 \Rightarrow y' = 0$$

$$y = \frac{\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$$

$$y = \frac{\pi}{2} \Rightarrow \sin y = 1, \quad y' = 1$$

$$y = \frac{3\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$$

$$y = \pi \quad \sin y = 0, \quad y' = 0$$

$$y = \frac{5\pi}{4} \quad \sin y = -\frac{1}{\sqrt{2}} \approx -0.7, \quad y' \approx -0.7$$

Equilibrium solution:

A constant solution such that $y' = 0$ for all t :

$$y' = f(t, y_0) = 0.$$

That is, $y(t) = y_0$ is a sol'n for all t .

Ex. 5(a) In Ex. 4, equilibrium sol'ns are $y=0, \pi, 2\pi, \text{etc.}$

Ex. 5(b) In $y' = t(y+1)^2$, the equilibrium sol'n is $y = -1$.

Ex. 5(c) In $y' = ty + 1$, there are no equilibrium solutions, because $y' = 0$ implies $y = -1/t$, which contradicts $y' = 0$ ($(-1/t)' = 1/t^2 \neq 0$).

Ex. 5(d) In $y' = y^2 + 1$ there are also no equilib. solutions because $(y^2 + 1) > 0$ for all y .

HW : Sec. 1.2 ## 5, 7, 8, 9, 13, 14, 21, 23

Hint for #23: Find a solution of this problem in a Calc. textbook (usually in Calc. III, under "velocity & acceleration").

~~#10~~ # 3

~~#~~ # 10 (Hint: proceed similarly to #23)

Sec. 1.3 5, 1, 2, 4, 10.