

Lecture 18 Particular sol'n of a
nonhomogeneous DE-2:
the method of variation of parameters.

① Brief review of DE-1 case

Solve $y' + p(t)y = g(t)$ (1)

given y_h : $y_h' + p(t)y_h = 0$. (2)

Solution: seek $y = y_h \cdot u(t) \Rightarrow$

$$(y_h \cdot u' + \underbrace{y_h' u}_{=0 \text{ by (2)}}) + p \cdot y_h \cdot u = g$$

$$\Rightarrow y_h \cdot u' = g \Rightarrow u' = g/y_h$$

$$\Rightarrow \text{solve for } u \Rightarrow y = y_h \cdot u. \quad \checkmark$$

② Formula for DE-2 case

Solve $y'' + p(t)y' + q(t)y = g(t)$ (3)

given that $\{y_1, y_2\}$ form a FS of the
homogeneous DE-2

$$y'' + py' + qy = 0. \quad (4)$$

Use a similar "trick": seek

Trick #1

$$\boxed{y_p(t) = y_1 \cdot u_1(t) + y_2 \cdot u_2(t)}, \quad (5)$$

where $u_1(t), u_2(t)$ are TBD.

$$y_p' = (\underline{y_1' u_1} + y_1 u_1') + (\underline{y_2' u_2} + y_2 u_2')$$

rearrange

$$\stackrel{\text{rearrange}}{=} [\underline{y_1' u_1} + \underline{y_2' u_2}] + [y_1 u_1' + y_2 u_2']$$

new

Before we compute y_p'' , use Trick #2:

Require:

$$y_1 u_1' + y_2 u_2' = 0 \quad (6)$$

Continue ^{while} using (6):

$$y_p' = [y_1' u_1 + y_2' u_2] \Rightarrow$$

$$y_p'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'$$

Substitute into (3):

$$\begin{aligned} & (y_1'' u_1 + y_1' u_1' + \underline{y_2'' u_2} + y_2' u_2') + p \cdot (\underline{y_1' u_1} + \underline{y_2' u_2}) \\ & + q \cdot (\underline{y_1 u_1} + \underline{y_2 u_2}) = g(t) \end{aligned}$$

Rearrange:

$$\begin{aligned} & \overset{0}{\cancel{(y_1'' + p y_1' + q y_1)} \cdot u_1} + \overset{0}{\cancel{(y_2'' + p y_2' + q y_2)} \cdot u_2} \\ & + [y_1' u_1' + y_2' u_2'] = g \end{aligned}$$

$$\Rightarrow \boxed{y_1' u_1' + y_2' u_2' = g} \quad (7)$$

Note 1: Had we not required (6), then y'' would have had terms like $y_1 u_1''$ & $y_2 u_2''$, and (7) would not have had its simple form.

However, a systematic explanation for trick #2 will be given only in Chap. 4.

Continuing with the solution:

$$(6): \quad y_1 u_1' + y_2 u_2' = 0 \quad | \cdot y_2'$$

$$(7): \quad y_1' u_1' + y_2' u_2' = g \quad | \cdot y_2$$

This is a linear system for $\{u_1', u_2'\}$:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \quad (8)$$

Its solution exists for any g because

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv W(t) \neq 0, \text{ because } \{y_1, y_2\} \text{ is a FS of sol'ns.}$$

Solve (6) & (7).

E.g., (6) $\cdot y_2'$ - (7) $\cdot y_2$ (see above) yields:

$$(y_1 y_2' - y_1' y_2) u_1' + (\cancel{y_2 y_2'} - \cancel{y_2' y_2}) u_2' = 0 - g \cdot y_2$$

$$\Rightarrow \boxed{u_1' = \frac{-g \cdot y_2}{W(t)}} \quad (9a)$$

Similarly:

$$\boxed{u_2' = \frac{g \cdot y_1}{W(t)}} \quad (9b)$$

Integrating:

$$u_1(t) = - \int^t \frac{g(s) y_2(s)}{W(s)} ds \quad (10a)$$

$$u_2(t) = \int^t \frac{g(s) y_1(s)}{W(s)} ds \quad (10b)$$

Combine (10) & (5): **MUST MEMORIZE**

$$y_p(t) = -y_1(t) \int^t \frac{g(s) y_2(s)}{W(s)} ds + y_2(t) \int^t \frac{g(s) y_1(s)}{W(s)} ds$$

(11)

Note 2: The upper limit "t" above indicates that after the integration, we should set the variable $s = t$.

The lower limit is not important because it will only shift the integral by a constant. Then sol'n (11) will shift by:

$$a_1 y_1 + a_2 y_2 = y_c(t) \leftarrow \text{sol'n of homogeneous DE,}$$

and changing y_p by some $y_c(t)$ does not affect the general solution

$$y(t) = y_p(t) + C_1 y_1 + C_2 y_2. \quad (12)$$

③ Examples

Ex. 1 $y'' - y' - 2y = Ge^{2t}$
Find y_p .

Note 1: We solved this in Lec. 17 by a substitution

$$y_p = A \cdot t e^{2t} \quad (13)$$

(because $y_h = e^{2t}$ is a sol'n of the homogeneous problem).

However, this substitution was postulated but not derived.

Now we will derive (13).

Sol'n:

1) Ex. 4 in Lec. 17 \Rightarrow

$$y_1 = e^{-t}, \quad y_2 = e^{2t}; \quad W(t) = 3e^t \quad (\text{p. 124 of book})$$

We need $u_1(t), u_2(t)$,

let's compute $u_2(t)$ and show that term

$y_2 \circ u_2$
agrees with (13)

Note 2: In HW 18 you may explore what happens to term $y_1 u_1$.

2) From formula (10b):

$$u_2(t) = \int^t \frac{g(s) y_1(s)}{W(s)} ds = \int^t \frac{G e^{2s} \cdot e^{-s}}{3e^s} ds =$$

18-6

$$= \int_{t_0}^t \frac{G}{3} ds = \frac{G}{3} s \Big|_{t_0}^t = \frac{G}{3} t - \underbrace{\frac{G}{3} t_0}_{= \text{const}}$$

\uparrow some const

Thus,

$$y_2 \cdot u_2 = y_2 \cdot \left(\frac{G}{3} t - \frac{G}{3} t_0 \right) =$$

$$= \underbrace{\frac{G}{3} t e^{2t}}_{\text{same as in Ex. 4 of Lec. 17}} + \underbrace{\left(-\frac{G}{3} t_0 \right) \cdot e^{2t}}_{\text{some const}}$$

same as in Ex. 4
of Lec. 17

part of the
homogeneous
sol'n.

Ex. 2 Find the general sol'n of

$$t^2 y'' - t y' + y = t$$

Sol'n:

1) Always put into form

coefficient = 1 $\rightarrow y'' + p y' + q y = g,$

because previous formulas were derived for this form (i.e. when the coefficient of y'' is 1).

So:

$$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = \frac{1}{t}$$

$$\Rightarrow g(t) = 1/t.$$

2) In #15 of HW13 you showed that

$$y_1 = t, \quad y_2 = t \ln t$$

form a fundamental set of sol'n's for the homogeneous version of this DE.

$$W(t) = \begin{vmatrix} t & t \ln t \\ 1 & (\ln t) + 1 \end{vmatrix} = t \ln t + t - t \ln t = t \quad (\neq 0 \text{ for } t \neq 0).$$

$$3) (10a) \Rightarrow u_1(t) = - \int \frac{g(s) y_2(s)}{W(s)} ds$$

$$= - \int \frac{\frac{1}{s} \cdot s \ln s}{s} ds = - \int \frac{\ln s}{s} ds \quad \leftarrow dv \quad \begin{array}{l} v = \ln s \\ dv = ds/s \end{array}$$

$$= - \int_{s=t}^t v \cdot dv = - \frac{v^2}{2} \Big|_{s=t}^t = - \frac{(\ln s)^2}{2} \Big|_{s=t}^t = - \frac{(\ln t)^2}{2}.$$

$$(10b) \Rightarrow u_2 = \int \frac{g(s) y_1(s)}{W(s)} ds = \int \frac{\frac{1}{s} \cdot s \cdot ds}{s} = \int \frac{ds}{s}$$

$$= \ln|s| \Big|_{s=t}^t = \ln|t| \xrightarrow[\substack{\text{assume} \\ t > 0}]{\quad} \ln t.$$

$$4) y_p = - \frac{(\ln t)^2}{2} \cdot t + \ln t \cdot t \ln t = \frac{1}{2} t (\ln t)^2$$

General sol'n:

$$y(t) = c_1 t + c_2 t \ln t + \frac{1}{2} t (\ln t)^2.$$

Ex. 3. Find y_p and then solve the IVP:

$$y'' + \omega^2 y = g(t)$$

$$y(0) = y_0, \quad y'(0) = y_0'$$

Sol'n: 1) Lec. 10 (also Lec. 14) \Rightarrow

$y_1 = \cos \omega t$, $y_2 = \sin \omega t$ form a FS;

$$W(t) = \omega (= \text{const}).$$

$$2) u_1 = - \int_a^t \frac{g(s) y_2(s)}{W(s)} ds$$

← some constant.

Let us choose (a) so that $y_p(0) = 0$;
then c_1, c_2 in $y_c(t)$ will be chosen
to satisfy the IC.

details
will be
given later

Then $a=0$, because $\int_0^0 (\text{anything}) = 0$.

$$u_1(t) = - \int_0^t \frac{1}{\omega} \cdot g(s) \cdot \sin(\omega s) ds \equiv - \frac{1}{\omega} \int_0^t g(s) \sin(\omega s) ds$$

$$u_2(t) = \int_0^t \frac{1}{\omega} g(s) \cos(\omega s) ds$$

$$(*) \quad y_p(t) = \frac{1}{\omega} \left[-\cos(\omega t) \cdot \int_0^t g(s) \sin(\omega s) ds + \sin(\omega t) \cdot \int_0^t g(s) \cos(\omega s) ds \right]$$

$$= \frac{1}{\omega} \int_0^t g(s) \left[\sin(\omega t) \cos(\omega s) - \sin(\omega s) \cos(\omega t) \right] ds$$

trig identity \Downarrow

$$\equiv \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s]) ds$$

General: $y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s]) ds$.

3) $y(0) = y_0 \Rightarrow$

$c_1 \cdot 1 + c_2 \cdot 0 + y_p(0) = y_0 \Rightarrow c_1 = y_0$.
0 (see above)

$y'(0) = y_0' \Rightarrow$

$c_1 \cdot (-\omega) \cdot 0 + c_2 \cdot \omega \cdot 1 + y_p'(0) = y_0'$.

To find $y_p'(0)$, one could have differentiated Eq. (*) on previous page. For that, one needs:

- product rule, and
- Fund. thm. of Calculus.

However, this is tedious.

Fortunately, there is another "Rule" of differentiation, which works here, as well as in many other circumstances. It is based on a composition of two familiar rules.

• Familiar Rule 1:

$\frac{d}{dt} \int_a^b f(t,s) ds = \int_a^b \frac{\partial f(t,s)}{\partial t} ds$

• Familiar Rule 2 (Fund. Thm. of Calculus):

$\frac{d}{dt} \int_a^t h(s) ds = h(t)$.

NEW RULE:

MUST MEMO-RIZE

$\frac{d}{dt} \int_a^t f(t,s) ds = \int_a^t \frac{\partial f(t,s)}{\partial t} ds + f(t,t)$ (14)

from F. Rule 1 from F. Rule 2

Apply this NEW RULE to $y_p'(t)$:

$$y_p'(t) = \frac{1}{\omega} \int_0^t g(s) \cdot \omega \cdot \cos(\omega[t-s]) ds + \frac{1}{\omega} \cdot g(t) \cdot \sin(\omega[t-t])$$

$$= \int_0^t g(s) \cos(\omega[t-s]) ds.$$

Thus $y_p'(0) = \int_0^0 \dots = 0.$

Note 1: With the $y_p(t) = \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s]) ds$,
one has convenient conditions:

$$y_p(0) = 0 \text{ and } y_p'(0) = 0.$$

Thus, coming back to the top of p. 18-9:

$$y'(0) = y_0' \Rightarrow 0 + c_2 \cdot \omega + 0 = y_0' \Rightarrow c_2 = \frac{y_0'}{\omega}.$$

Thus the solution of the IVP is:

$$y(t) = y_0 \cos \omega t + \frac{y_0'}{\omega} \sin \omega t + \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s]) ds.$$

Note 2: An integral of the form

$$\int_0^t f(t-s) g(s) ds$$

is called a convolution of f and g .

We will (hopefully) encounter it again
in Chap. 5.

HW, Sec. 3.9

- 1 \leftarrow ($g = \text{const}$); compare to method of Lec. 15.16.
 3, 4, 5, 7, 10, 13 \leftarrow find $y_c, y_p, y_{\text{general}}$.
 16, 18 \leftarrow from ^{given} form $y = y_c + \int K(t-s)g(s)ds$ find
 α, β, y_0, y_0' in $y'' + \alpha y' + \beta y = g$.

WP: Solve IVP $y'' + \omega^2 y = G \sin \omega t$ using method of
 Ex. 3 from Notes. Hint: use $\sin a \cdot \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$.

Note for #4: Can do both w/ Mathematca. (???)
 If you want to do them by hand, use subst. $e^t = u$;
 then ~~one~~ of them will require a part. fr. expansion \rightarrow look it up!

Note for #5: a) Compare the so-obtained particular sol'n
 with that obtained by method from Lec. 17.
 How much do they differ by?

1) This will show you what happens to term $y_1 u_1$ (see Ex. 1 in class).

Note for #7: Again, compare your y_p with that from method
 of und. coeff.

Note for #13: 1) Denote $(t-1) = r$; relate dy/dt and dy/dr ,
 and solve the entire problem in terms of $y(r)$.
 2) will need to find y_2 by method of Lec. 13.

Note for #18: Use the form $y = c_1 y_1 + c_2 y_2 + y_1 u_1 + y_2 u_2$
 and the expressions for u_1, u_2 .
 Answer: $\alpha = \beta = 0, y_0 = 0, y_0' = 1$.