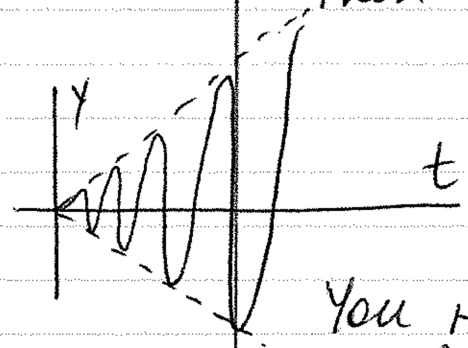
Lecture 19. Resonance in undamped and damped linear oscillator models

① Exact resonance: review of known results

In Lec. 17 (Ex. 5) we showed that when the external force (proportional to function $g(t)$) has the same frequency ω as the oscillator's own frequency:

$$y'' + \underbrace{\omega^2}_{\text{oscillator's own frequency}} y = G \cdot \sin \underbrace{\omega}_{\text{frequency of external force}} t \qquad (1)$$

then the particular solution grows with time:



$$y_p = -\frac{G}{2\omega} \cdot \underbrace{t}_{\text{growing "envelope"}} \cos \omega t \qquad (2)$$

You reobtained this (~~or~~ very similar) result in the HW #18.

This phenomenon, namely:

$$\begin{pmatrix} \text{if frequency of external force} \\ = \\ \text{own frequency of oscillator} \end{pmatrix} \Rightarrow \begin{pmatrix} \text{undamped} \\ \text{oscillations} \\ \text{grow with time} \end{pmatrix}$$

is called <u>the resonance</u>.

This is a very important phenomenon in diverse areas of science and engineering.

When (frequency of external force) = (own frequency of oscillator),

we say that the force is <u>at resonance</u>.

Below we'll explore situations when the force is <u>near resonance but not at resonance</u>, or when a damping is present in the oscillator.

(2) <u>Non-resonant force</u>: <u>phenomenon of "beats"</u>.

We will first consider an oscillator w/o damping:

$$y'' + \underbrace{\omega_0}_{\text{own frequency of oscillator}}{}^2\, y = F\cdot\cos \underbrace{\omega_1}_{\text{external force frequency}} t \qquad (3)$$

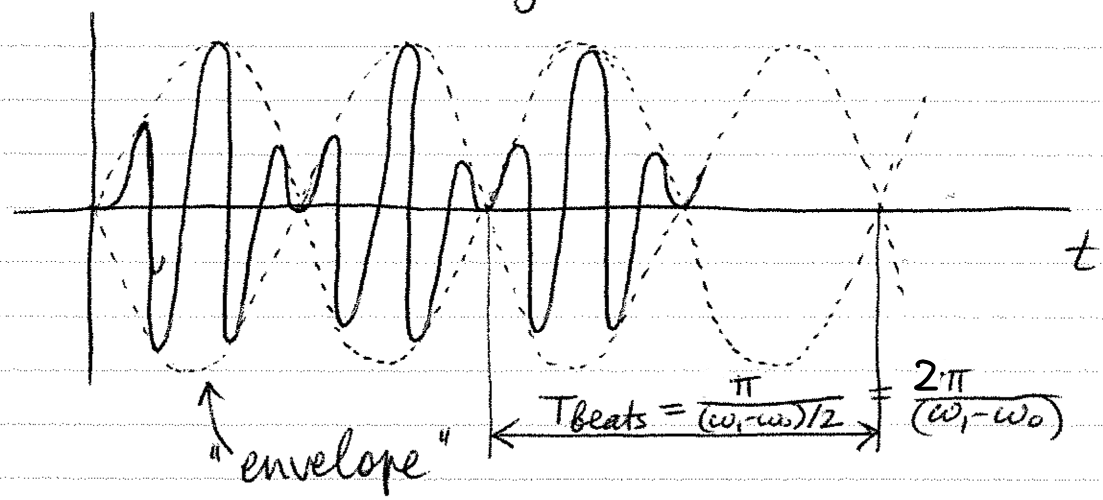
For simplicity, let us assume zero IC:

$$y(0) = 0, \quad y'(0) = 0. \qquad (4)$$

By the method of Lec. 17, we have (see Word Problem 3 in HW17):

$$y(t) = \frac{-F}{\omega_0^2 - \omega_1^2} \left(\cos\omega_0 t - \cos\omega_1 t\right) \qquad (5)$$

It is shown in Fig. 3.14 in Sec. 3.10:



It is a sinusoidal function modulated by an envelope. To understand this shape, use a trig. identity:

$$\cos(\omega_0 t) - \cos(\omega_1 t) = \underbrace{2\sin\left(\frac{\omega_1-\omega_0}{2}t\right)}_{\text{envelope}} \cdot \underbrace{\sin\left(\frac{\omega_1+\omega_0}{2}t\right)}_{\text{oscillations inside the envelope.}}$$

$$\left(\cos(a-b) - \cos(a+b) = 2\sin a \cdot \sin b\right)$$

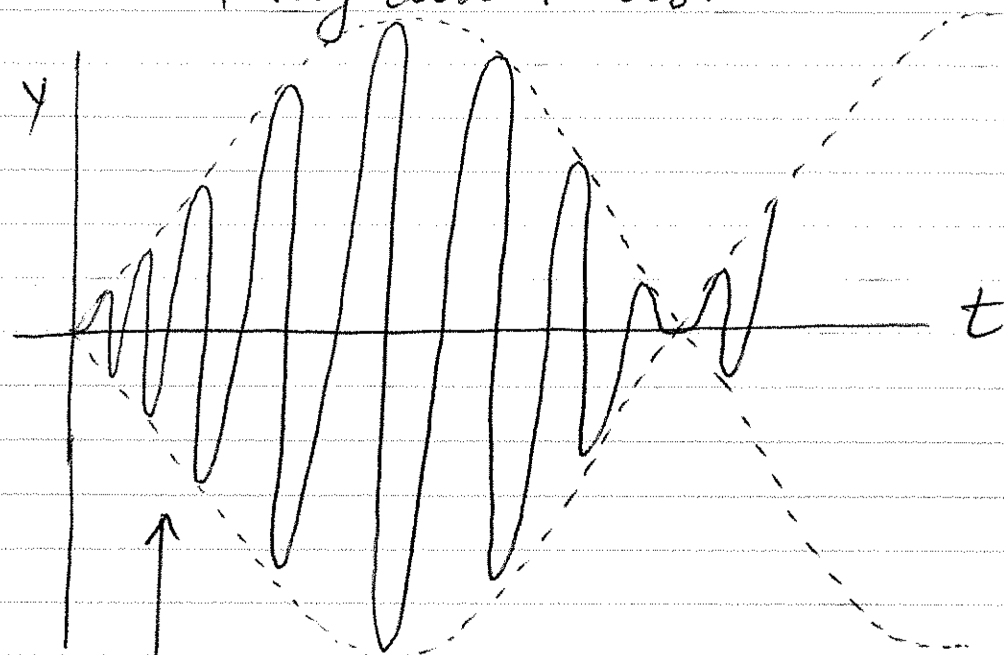
Note 1 As $\omega_1 \to \omega_0$, two things occur:

1) Amplitude of the oscillations grows as:

$$R \sim \frac{F}{|\omega_1^2 - \omega_0^2|} \to \infty \quad \text{as } \omega_1 \to \omega_0.$$

2) Period of the beats increases as $\frac{2\pi}{|\omega_1 - \omega_0|}$.

For ω_1 very close to ω_0:



Thus, initially, we have growing osillations, just as in the exact resonance.

<u>Note 2</u> We can see by the L'Hospital's Rule that as $\omega_1 \to \omega_0$, the solution (5) $\to \frac{F}{2\omega_0} t \cdot \sin\omega_0 t$ (WP3 in HW17):

$$\lim_{\omega_1\to\omega_0} (5) = \lim_{\omega_1\to\omega_0} \frac{-F}{\omega_1^2-\omega_0^2}(\cos\omega_1 t - \cos\omega_0)$$

$$= \lim_{\omega_1\to\omega_0}\left(\frac{-F}{\omega_1+\omega_0}\right) \cdot \frac{\cos(\omega_1 t)-\cos(\omega_0 t)}{\omega_1-\omega_0} =$$

$$= \lim_{\omega_1\to\omega_0} \frac{-F}{\omega_1+\omega_0} \cdot \lim_{\omega_1\to\omega_0} \frac{\cos(\omega_1 t)-\cos(\omega_0 t)}{\omega_1 - \omega_0} =$$

$$= \frac{-F}{2\omega_0} \cdot \frac{d}{d\omega_1}(\cos\omega_1 t)\Big|_{\omega_1=\omega_0}$$

$$= \frac{\cancel{t}F}{2\omega_0}\cdot\left(\cancel{t}\,t\cdot\sin\omega_0 t\right) = \frac{F}{2\omega_0}\cdot t\cdot\sin\omega_0 t.$$ ✓

<u>Note 3</u> Of course, physically, a solution describing oscillations whose amplitude $\to\infty$ as $t\to\infty$ makes sense only up to some finite t_{max}. Beyond it, the assumptions of the model get (gradually) violated, and the amplitude saturates at some large but finite value. We'll examine this now, when <u>small damping</u> arrests the growth of amplitude.

3 <u>Effect of damping in the presence of resonant force.</u>

In the notations of Lec. 15:

same!

$$\underbrace{y'' + 2\alpha y' + \omega_0^2 y}_{} = F\cos\omega_0 t \qquad (6)$$

Damped linear oscillator with own frequency ω_0. (In book, "α" is denoted as "δ" (delta).)

<u>"resonant"</u> force (same frequency as the osillator's)

Assume zero initial conditions:

$$y(0) = 0, \quad y'(0) = 0. \qquad (7)$$

By Lec. 16,

$$y = y_p + y_c$$

$$= \underbrace{(A_1 \sin\omega_0 t + B_1 \cos\omega_0 t)}_{\text{by Lec. 17, since } \sin\omega_0 t \text{ \& } \cos\omega_0 t \text{ are not homogeneous sol'ns}} + \underbrace{e^{-\alpha t}(A_2 \sin\beta t + B_2 \cos\beta t)}_{\text{by Lec. 15, where } \beta = \sqrt{\omega_0^2 - \alpha^2}}.$$

In a HW problem you will verify that $B_1 = B_2 = 0$ for the IC (7), and that

$$y = \frac{F}{2\alpha}\Big(\underbrace{\frac{\sin\omega_0 t}{\omega_0}}_{\text{from } y_p} - \underbrace{e^{-\alpha t} \cdot \frac{\sin\beta t}{\beta}}_{\text{from } y_c} \Big) \qquad (8)$$

WP 2 of HW17

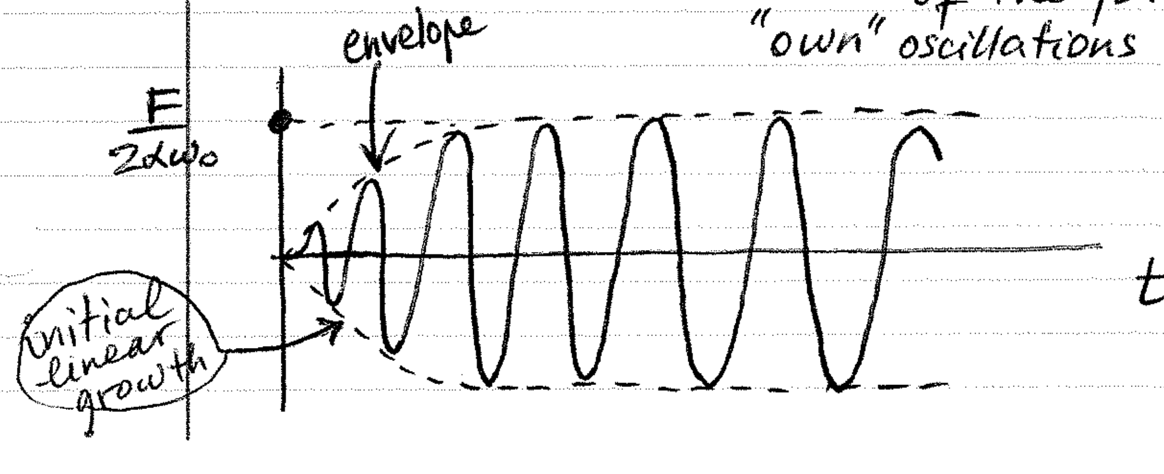
Key property of this solution:

As $t \to +\infty$, $y_c \to 0$ as $e^{-\alpha t} \to 0$, $\Rightarrow$

$$y(t \to +\infty) \to \frac{F}{2\alpha\omega_0} \sin\omega_0 t \qquad (9)$$

amplitude

only oscillations with the frequency of the force remain; "own" oscillations die out.



(4) Damping and non-resonant force

$$y'' + 2\alpha y' + \omega_0^2 y = F \cos \omega_1 t \quad (10)$$

$\omega_1 \neq \omega_0$.

The form of the solution is still

$$y = y_p + y_c,$$

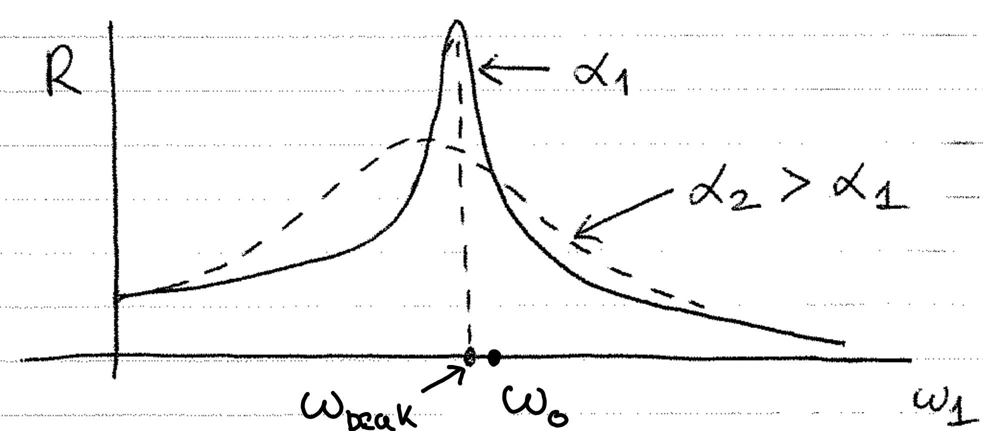
but the coefficients are much more cumbersome: see Eq. (12b) on p. 180. The plot is also not as simple as for $\omega_1 = \omega_0$ (see p. 181).

Essential feature:

As $t \to +\infty$, $y(t) \to y_p = R \cdot \cos(\omega_1 t - \delta)$,

since $y_c \sim e^{-\alpha t} \to 0$

$$R = \frac{F}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + (2\alpha\omega_1)^2}} \quad (11)$$



Essential features of non-resonant solution in presence of damping:

1) For $\alpha_1 < \alpha_2$, solution has larger amplitude for $\omega_1 \approx \omega_0$,

2) but a smaller amplitude for ω_1 far from ω_0.

("The resonance is narrower when damping is smaller.")

3) The frequency of the peak is $\approx \sqrt{\omega_0^2 - 2\alpha^2}$, i.e. is left-shifted.
The greater the damping, the greater the left-shift of the peak.

HW: Sec. 3.10 ##
2, 3, 4, 6 ← no damping
resonance
10(a,c), 12(b,c) [Accept Eq.(12b) on p.180 as given],
9 ← res. force in presence of damping.

#2: $\frac{t}{10} \sin 10t$