Lec. 2. Linear DEs

(1) Linear & nonlinear DEs. Linear homogeneous & non-homogeneous DEs

Def: DE $y' = f(t,y)$

is called linear if $f(t,y) = \underbrace{a(t)}\, y + \underbrace{b(t)}$

arbitrary functions of t

just y (i.e., no y^2, $1/y$, $\sin y$, etc.)

Alternative form used more frequently in this course:

$$y' + p(t)y = g(t) \qquad (1)$$

Def: If in (1), $g(t) = 0$ for all t, the DE is called homogeneous.
If $g(t) \neq 0$ for some t, $\Rightarrow$ non-homogeneous.

Ex. 1 Classify equations as linear or nonlinear. If they are linear, further classify as homogeneous or non-homogeneous.

(a) $y' = ty^2 \leftarrow$ nonlinear

(b) $y' = t^2 y \leftarrow$ linear ($p = -t^2$, $g = 0$) homogeneous

(c) $(y')^2 = ty \Rightarrow$

$y' = \pm\sqrt{ty}$ ← nonlinear

(d) $(\cos t)\, y' + e^t \cdot y = \sin t \Rightarrow$

$y' + \underbrace{\frac{e^t}{\cos t}}_{p(t)} \cdot y = \underbrace{\tan t}_{g(t) \neq 0}$ ← linear

← non-homogeneous

② Solution of linear homogeneous DE.

$$y' + p(t)\, y = 0, \quad \text{or}$$

$$y' = -p(t)\, y. \qquad (2)$$

Let $\underline{P}(t) = \int p(t)\, dt$, or

$$\underline{P}'(t) = p(t). \qquad (3)$$

The solution of (2) is based on two observations.

1) $\left(e^{-P(t)}\right)' \overset{\text{Chain Rule}}{=} \left(-\underline{P}(t)\right)' \cdot e^{-P(t)} \overset{(3)}{=} -p(t)\, e^{-P(t)}$.

So denote $z(t) = e^{-P(t)}$; then:

$z' = -p(t)\, z$. ← Looks like (2).

So we've found one sol'n of (2).

2) To find the general sol'n, note that

$$y = C \cdot z, \quad \text{where } C = \text{any const},$$

also satisfies (2):

$$C \cdot (z' = -pz)$$

$$Cz' = -Cpz$$

$$(Cz)' = -p(Cz).$$

Thus,
$$\boxed{y = Ce^{-P(t)}} \qquad (4)$$

is the general sol'n of (2).

Note: Since we've, basically, guessed the sol'n $z = e^{-P(t)}$, we cannot yet guarantee that we haven't missed some other sol'ns.
We will prove this ~~[illegible]~~ in Lec. 5.
So for now we take on faith that (4) is the most general sol'n of (2).

3 Sol'n of linear nonhomogeneous DE.

Method 1 Integrating factor

1) Observation about the homogeneous DE (2).

$$e^{P} \cdot (y' + py = 0)$$

$$e^{P} \cdot y' + \underbrace{(e^{P} \cdot p)}_{(e^{P})'} \cdot y = 0$$

Product Rule: $(e^{P}\cdot y)' = 0$

$$e^{P}\cdot y = C$$

$$y = Ce^{-P(t)}.\quad \text{(Agrees with (4).)}$$

2) Apply this idea to the nonhomogeneous DE:

$$y' + py = g \qquad (5)$$

$$e^{P}\cdot(y' + py = g)$$

same as above $\rightarrow (e^{P}y)' = e^{P}g$

$$e^{P}y = \int e^{P(t)}g(t)\,dt$$

$$y = e^{-P(t)}\int e^{P(t)}g(t)\,dt \qquad (6)$$

is the solution of (5).

<u>Note 1</u> The indefinite integral in (6) contains, by definition, an arbitrary constant.

Let us explicitly account for this by rewriting:

$$\boxed{y = e^{-P(t)}\Big(\underbrace{\int e^{P(t)}g(t)\,dt}_{\substack{\text{some particular}\\ \text{value of indefinite }\int}} + C\Big)}. \qquad (7)$$

! MUST MEMORIZE

e.g. $\int (t-1)\,dt = \frac{(t-1)^2}{2}$.

<u>Note 2</u> It may look like $e^{-P(t)}$ outside the $\int$ and $e^{P(t)}$ inside the $\int$ will cancel.
However, this is <u>not</u> so.

<u>Ex. 2</u> Let $p(t) = -\frac{2t}{1+t^2}$, $g(t) = 2t$.

Let us set $C = 0$ in (7) for now.

$$P(t) = \int p(t)dt = -\int \frac{2t}{1+t^2}dt \Big|_{u=1+t^2} = -\ln(1+t^2).$$

$$e^{P(t)} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

$$e^{-P(t)} = (1+t^2).$$

$$y = \underbrace{(1+t^2)}_{e^{-P}} \int \underbrace{\frac{1}{1+t^2}}_{e^{P}} \cdot \underbrace{2t}_{g} \cdot dt = \overbrace{(1+t^2) \cdot \ln(1+t^2)}^{\text{as above}}$$

This is <u>not</u> the same as

$$\cancel{e^{-P}} \int \cancel{e^{P}} \cdot g \cdot dt = \int 2t\, dt = t^2.$$

<u>Note 3</u> This potentially confusing notation (which makes us want to cancel $e^{-P(t)}$ and $e^{P(t)}$) is <u>eliminated</u> when we solve an <u>IVP</u> and use correct notations.

<u>IVP</u>:
$$\begin{aligned} y' + p(t)y &= g(t) \\ y(t_0) &= y_0 \end{aligned} \qquad (8)$$

Its solution is:

$$\boxed{\begin{aligned} y(t) &= e^{-P(t)}\left(\int_{t_0}^{t} e^{P(t_1)} g(t_1)\,dt_1 + y_0\right) & (9a)\\ P(t) &= \int_{t_0}^{t} p(t_2)\,dt_2 & (9b)\end{aligned}}$$

(Indeed, (9) has the form (7) and also satisfies the initial condition $y(t_0) = y_0$, because

$$P(t_0) = \int_{t_0}^{t_0} \ldots = 0, \quad e^0 = 1,$$

$$\text{and} \int_{t_0}^{t_0} e^{P(t_1)} g(t_1)\,dt_1 = 0 .)$$

So for solving IVPs (8) we will use solution in the form of (9).

Ex. 3 Solve IVP

$$y' - \frac{2t}{1+t^2}\,y = 2t$$

$$y(1) = 3$$

Sol'n: 1) $P(t) = -\int_{1}^{t} \frac{2t_2}{1+t_2^2}\,dt_2 \overset{\text{as in Ex. 2}}{=} -\ln(1+t_2^2)\Big|_1^t$

Use the given $t_0 = 1$.

$$= -\left(\ln(1+t^2) - \ln(1+1^2)\right) = \ln 2 - \ln(1+t^2).$$

2) $e^{P(t)} = e^{\ln 2 - \ln(1+t^2)} = \dfrac{e^{\ln 2}}{e^{\ln(1+t^2)}} = \dfrac{2}{1+t^2}$.

$$e^{-P(t)} = \frac{1+t^2}{2}.$$

3) $\displaystyle\int_{t_0}^{t} e^{P(t_1)} \cdot g(t_1)\,dt_1 = \int_{1}^{t} \frac{2}{1+t_1^2} \cdot 2t_1\,dt_1 \overset{\text{Ex. 2}}{=}$

$$= 2\ln(1+t_1^2)\Big|_1^t = 2(\ln(1+t^2)-\ln 2).$$

4) Put together:

$$y(t) = \frac{1+t^2}{2}\cdot\Big(2(\ln(1+t^2)-\ln 2) + \overset{y_0}{3}\Big).$$

Note: It is not always a good idea to try to simplify an answer like this because then we'll lose the easy way to see that $y(t_0) = y_0$ indeed.

Method 2 ← preferred method.

Variation of parameter

We will obtain the same formula, (6) or (7). However, the method is more general in that it can be extended to systems of several ODEs (Chap. 4).

Recall: $y_h = C_0 e^{-P(t)}$, $C_0 = \text{const}$

solves the homogeneous DE: $y_h' + p(t) y_h = 0$.

Want: Solve $y' + py = g$.

Trick: Seek $y = C(t) e^{-P(t)}$ (10)

Allow parameter C to vary (not be const.)

Substitute (10) into (5):

$$y' + py = g, \quad y = C(t)\,y_h$$

$$(Cy_h)' + pCy_h = g$$

$$C'y_h + \underbrace{Cy_h' + Cpy_h}_{C(y_h'+py_h)\to 0 \text{ by the homogeneous DE}} = g$$

$$C'y_h = g \Rightarrow C' = \frac{g}{y_h}$$

(choose $C_0 = 1$)

$$C' = \frac{g}{e^{-P(t)}} = e^{P(t)}g(t),$$

$$\Rightarrow \quad C = \int e^{P(t)}g(t)\,dt.$$

Thus, (10) becomes:

$$y(t) = e^{-P(t)} \int e^{P(t)} g(t)\,dt. \leftarrow \text{same as (6).}$$

Again: We'll use the same method (but not method 1) in Chap. 4.

④ Important special cases of solutions of homogeneous linear DEs.

Ex. 4 Linear homogeneous DE with zero initial condition has only the trivial solution ($y(t) \equiv 0$ for all t).

<u>Proof</u>: $y' + p(t)y = 0, \quad y(t_0) = 0$ zero i.c.

By (4),

$$y(t) = C \cdot e^{-P(t)}.$$

<u>@ $t = t_0$</u>: $0 = C \cdot e^{-P(t)} \Rightarrow C = 0$

$$\Rightarrow y(t) = 0.$$

<u>Note:</u> In this case, $y = 0$ is the <u>equilibrium</u> solution.

<u>Ex. 5</u> <u>Exponential solutions</u>

$p(t) = -a \ (= \text{const})$.

$$\boxed{y' = ay, \quad y(t_0) = y_0.} \qquad (11a)$$

By (9) with $g(t) = 0$:

$$P(t) = \int_{t_0}^{t} (-a)\,dt_1 = -a(t - t_0).$$

$$y(t) = e^{-P(t)} \left(\int_{t_0}^{t} e^{P(t_1)} \cancelto{0}{g(t_1)}\,dt_1 + y_0 \right)$$

$$= y_0 e^{a(t-t_0)}.$$

Thus, $\boxed{y(t) = y_0 e^{a(t-t_0)}}$ (11b)

solves (11a).

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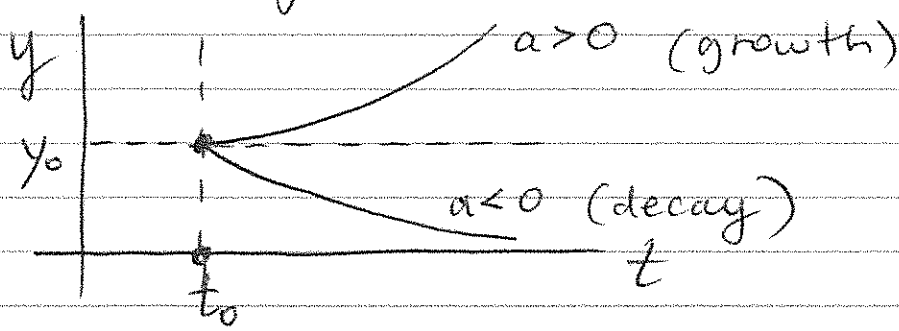
<u>Note:</u> Depending on the sign of a, the solution either exponentially grows or exponentially decays!



<u>Ex, 6</u> <u>Rational solutions</u> : $p(t) = \frac{-a}{t}$.

$P(t) = \int \frac{-a}{t}\,dt = -a\ln|t|$ (let $t>0$ for concreteness)

$e^{-P(t)} = e^{a\ln t} = \left(e^{\ln t}\right)^a$
$= t^a$.

Then $\boxed{y = C\cdot e^{-P(t)} = C\cdot t^a}$

is the general solution of $y' - \frac{a}{t}y = 0$.

(do <u>not</u> need to memorize)

<u>Observation from Exs. 5 & 6</u> : $y' + p(t)y = 0$

$\underline{p(t)<0}$ (const. or not) $\Rightarrow y \to \infty$ as $t \to \infty$

$\underline{p(t)>0}$ (const. or not) $\Rightarrow y \to 0$ as $t \to \infty$.

This is true for <u>any</u> $p(t)$, not just const or $\frac{\text{const}}{t}$.

5 Special nonhomogeneous linear DE which is reducible to homogeneous DE.

$$y' = ay + b, \quad a, b = \text{const} \ (a \neq 0).$$

$$y' = a\left(y + \frac{b}{a}\right) \leftarrow \text{const}$$

$$\underbrace{\left(y + \frac{b}{a}\right)'}_{z} = a\underbrace{\left(y + \frac{b}{a}\right)}_{z}$$

$$z' = az \Rightarrow z = Ce^{at} \Rightarrow$$

$$\boxed{y = -\frac{b}{a} + Ce^{at}}.$$

Qs: Suppose $a < 0$.
- What is y as $t \to \infty$?
- What is the equilibrium solution?

6 Discontinuous coefficients: Read p. 25 in book.

HW: Sec. 2.1 1, 3, 5, 6 [lin], 9 ← lin/nonlin, homog.?

Sec. 2.2 1, 3, 4, 7 ← general sol & sol'n of IVP
11, 14, 20, 21 ← general; 25 ← match DE to dir. field
27, 33 ← find parameters in DE given two solution points
28 ← match eq. to graph (idea: sign of y')
36, 37, 39 ← behavior at $t \to \infty$; 41 ← discont. coeff
29(c) ← topic 5; (29(a,b) done in class basically).

Sec. 2.9 Read p. 78-79 (Drag force, Case 1), #1, 4, 17 ← Hint: max height achieved for $v = 0$.

Sec. 2.3 Recall Newton's Cooling (Lec. 1, sub-Ex. 2(b)). Apply topic 5 above
Do #17, 21.