Lecture 20. General properties of higher-order linear DEs

So far we've studied 2nd-order linear DEs:

$$y'' + p_1(t)y' + p_0(y)y = g(t). \qquad (1)$$

We'll now state all of the facts that we've learned about (1), for a higher-order DE:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t). \qquad (2)$$

We'll also present some generalizations of the earlier results.

1 Existence & uniqueness

Thm. 3.5 Let in (2), $p_{n-1}(t), p_{n-2}(t), \ldots p_0(t)$, $g(t)$ be continuous functions on $t \in (a,b)$. Also, let $t_0 \in (a,b)$. Then the IVP consisting of DE (2) and IC

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots\ y^{(n-1)}(t_0) = y_0^{(n-1)} \qquad (3)$$

has a unique solution on (a,b).

2 Properties of homogeneous linear DEs

$\boxed{2a}$ Principle of linear superposition

Let $y_1(t), \dots y_r(t)$ be solutions of the <u>homogeneous</u> <u>linear</u> DE

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \dots + p_1(t) y' + p_0(t) y = 0. \quad (4)$$

In what follows:

- We'll assume that $p_{n-1}, \dots p_0$ are continuous on $t \in (a,b)$, so that a unique sol'n of IVP (4)+(3) exists.
- We'll drop the explicit dependence of p on t, i.e. will write p_0 for $p_0(t)$, etc.

Let $c_1, \dots c_r$ be arbitrary constants. Then

$$y = c_1 y_1 + \dots + c_r y_r$$

is also a sol'n of (4).

$\boxed{2b}$ Fundamental sets & Wronskians

Let $y_1, \dots y_n$ be solutions of (4). $\{y_1, \dots y_n\}$ is called a FS of sol'ns if <u>any</u> sol'n of the IVP (4)+(3) can be written as

$$y = c_1 y_1 + \dots + c_n y_n \qquad (5)$$

for <u>some</u> constants $c_1, \dots c_n$.

Let us now determine a condition which determines if a set of sol'ns is a FS or not.

Suppose $\{y_1, \dots y_n\}$ is <u>some</u> set of sol'ns of (4) and we are <u>trying</u> to construct a solution of IVP (4)+(3) in the form (5). Then:

<u>$y(t_0) = y_0$</u> $\Rightarrow$

$$y_1(t_0)c_1 + y_2(t_0)c_2 + \dots + y_n(t_0)c_n = y_0$$

<u>$y'(t_0) = y_0'$</u> $\Rightarrow$

$$y_1'(t_0)c_1 + y_2'(t_0)c_2 + \dots + y_n'(t_0)c_n = y_0'$$

$\vdots$

<u>$y^{(n-1)}(t_0) = y_0^{(n-1)}$</u> $\Rightarrow$

$$y_1^{(n-1)}(t_0)c_1 + y_2^{(n-1)}(t_0)c_2 + \dots y_n^{(n-1)}(t_0)c_n = y_0^{(n-1)}.$$

These n eqs can be written in <u>matrix form</u>:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & & \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix} \qquad (6)$$

This linear system has a unique sol'n for <u>any</u> values of IC, $y_0, y_0', \dots y_0^{(n-1)}$, (in other words, constants $c_1, \dots c_n$ in (5) can <u>always</u> be found) <u>iff</u> the matrix in (6) is non-singular. That is, its determinant $\neq 0$:

$$W(t) \equiv \begin{vmatrix} y_1(t) & y_2(t) & & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & & y_n^{(n-1)}(t) \end{vmatrix} \neq 0. \quad (7)$$

Wronskian.

Thus, $\{y_1, \dots y_n\}$ is a FS iff $W(t_0) \neq 0$ at some t_0.

As in Lecture 11 (for DE-2), one has the result:

If the conditions of Thm. 3.5 (Existence & Uniqueness) are satisfied, then:

- either $W(t) \equiv 0$ for <u>all</u> $t \in (a,b)$, or
- $W(t) \neq 0$ for <u>all</u> $t \in (a,b)$.

We'll now formulate a more precise result.

<u>Thm. 3.6 (Abel's thm.)</u>

Consider the homogeneous DE (4) and assume that conditions of Thm. 3.5 are satisfied. Then

$$W'(t) = -p_{n-1} \cdot W, \quad (8a)$$

i.e. (by Lec. 2)

$$W(t) = W(t_0)\, e^{-\int_{t_0}^{t} p_{n-1}(s)\,ds}. \quad (8b)$$

<u>Note</u>: Eq. (8b) implies our earlier statement that either $W(t) \equiv 0$ on (a,b) or $W(t) \neq 0$ <u>anywhere</u> on (a,b).
This is because $e^{(\text{any power})} \neq 0$.

We'll prove Abel's thm for $n=2$.
Steps of the general proof are similar.

<u>Auxiliary facts</u> about determinants
(stated for $n=2$, but true in general)

<u>Fact 1</u> $\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0$

(If two rows (or columns) in a matrix are the same, then det $= 0$.)

<u>Fact 2</u> $\begin{vmatrix} a_1 & a_2 \\ kb_1 & kb_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

<u>Fact 3</u> $\begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

<u>Fact "2+3"</u>

$$\begin{vmatrix} a_1 & a_2 \\ kb_1+mc_1 & kb_2+mc_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + m \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

<u>Fact 4</u> (product rule for determinants)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}' = \begin{vmatrix} a_1' & a_2' \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1' & b_2' \end{vmatrix}$$

(e.g., $(a_1 b_2)' = \underline{a_1' b_2} + \underline{a_1 b_2'}$)

Proof of Abel's Thm for $n=2$.

Let's rewrite DE (4) for $n=2$ as:

$$y'' = -p_1 y' - p_0 y \qquad (9)$$

Differentiate the Wronskian:

$$W' = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' \overset{F4}{=} \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0 \text{ by } F1} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

$$\overset{\text{use } (9)}{=} \begin{vmatrix} y_1 & y_2 \\ -p_1 y_1' - p_0 y_1 & -p_1 y_2' - p_0 y_2 \end{vmatrix}$$

$$\overset{F\text{"}2+3\text{"}}{=} -p_1 \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{W} - p_0 \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1 & y_2 \end{vmatrix}}_{=0 \text{ by } F1}$$

$$= -p_1 W.$$

Thus, for DE-2,

$$W' = -p_1 W. \qquad (10)$$

<u>Ex. 1</u> Suppose that for some FS of sol'ns of DE $y^{(4)} + y'' + y' = 0$ one has $W(1) = 2$. Find $W(10)$.

<u>Sol'n:</u> $y^{(4)} + 0 \cdot y''' + 1 \cdot y'' + 1 \cdot y' + 0 \cdot y = 0$

$\Rightarrow \underset{(n-1)}{p_3} = 0 \Rightarrow$ by Eq. (8b) $W(t) = W(1) \cdot e^{-\int_1^t 0\, ds}$

$\Rightarrow W(t) = W(1)$ for all t, $\Rightarrow W(10) = W(1) = 2$.

$\boxed{2c}$ FS always exist (a.k.a. How to find a FS)

Thm. 3.7 Suppose that conditions of Thm. 3.5 (Existence & Uniqueness) are satisfied for DE (4) on $t \in (a,b)$.
Then a FS of sol'ns exists for (4) on (a,b).

Proof (for $n=3$)

Consider 3 IVP's:

$$y_1''' + p_2 y_1'' + p_1 y_1' + p_0 y_1 = 0, \quad y_1(t_0) = 1, \ y_1'(t_0) = 0, \ y_1''(t_0) = 0$$

$$y_2''' + p_2 y_2'' + p_1 y_2' + p_0 y_2 = 0, \quad y_2(t_0) = 0, \ y_2'(t_0) = 1, \ y_2''(t_0) = 0$$

$$y_3''' + p_2 y_3'' + p_1 y_3' + p_0 y_3 = 0, \quad y_3(t_0) = 0, \ y_3'(t_0) = 0, \ y_3''(t_0) = 1.$$

- By Thm. 3.5, their solutions y_1, y_2, y_3 exist,
- Their Wronskian

$$W(t_0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

$\Rightarrow$ by Abel's Thm. $W(t) \neq 0$ on (a,b),

$\Rightarrow \{y_1, y_2, y_3\}$ form a FS.

2d Solutions in a FS are linearly independent

Def: Let $f_1(t), \dots f_r(t)$ be defined on $t \in (a,b)$. They are called linearly dependent

if there exist constants $k_1, \dots k_r$ s.t.

$$k_1 f_1(t) + k_2 f_2(t) + \dots + k_r f_r(t) = 0 \quad \text{for all } t \in (a,b), \tag{11}$$

and at least one of $k_1, \dots k_r \neq 0$.

Equivalent definitions:

Def - A Since at least one of k's $\neq 0$, suppose $k_1 \neq 0$. Then

$$f_1 = -\frac{k_2}{k_1} f_2 - \frac{k_3}{k_1} f_3 - \dots - \frac{k_r}{k_1} f_r$$

$$\equiv m_2 f_2 + m_3 f_3 + \dots m_r f_r \tag{12}$$

That is, one of the functions is a linear superposition of the others for all $t \in (a,b)$.

Ex. 2 (a) $f_1 = 2$, $f_2 = 3-t$, $f_3 = 5t+7$ is a linearly dependent set, because:

$$\underset{\substack{\uparrow \\ f_1}}{2} = \underset{\substack{\uparrow \\ m_2}}{\frac{5}{11}} \underset{\substack{\uparrow \\ f_2}}{(3-t)} + \underset{\substack{\uparrow \\ m_3}}{\frac{1}{11}} \underset{\substack{\uparrow \\ f_3}}{(5t+7)}$$

or $\quad -11 \cdot f_1 + 5 \cdot f_2 + 1 \cdot f_3 = 0$ for all t.

(b) $f_1 = 2$, $f_2 = 3-t$, $f_3 = 5t^2+7$

is not a lin. dep. set because one cannot express the <u>quadratic term in f_3</u> via the linear and constant terms in f_1, f_2.

<u>Def-B</u>. A set $\{f_1, \dots f_r\}$ is called <u>linearly independent</u> if (11) can hold for <u>all</u> $t \in (a,b)$ <u>only</u> for $K_1 = K_2 = \dots = K_r = 0$.

<u>Thm. 3.8</u> Let $\{y_1, \dots y_n\}$ be a FS of sol'ns of DE (4). Then this set is <u>lin. independent</u>.

(w/o proof; suggested proof in HW).

<u>Note</u>: In Lec. 11 we saw that if $\{y_1, y_2\}$ form a FS for a DE-2, they can be compared to two non-parallel vectors in the plane:

FS ($y_1 \nparallel y_2$) not a FS ($z_1 \parallel z_2$)

Two non-parallel vectors in the plane form <u>a basis</u>, meaning:

- they are linearly independent (by Def-A, this means that y_2 is <u>not</u> proportional to y_1, i.e. $y_2 \nparallel y_1$).
- <u>any</u> vector in the plane can be written as a lin. combination of y_1, y_2:

$$y = c_1 y_1 + c_2 y_2.$$

Now we see that for a n-th order linear homogeneous DE, solutions $y_1, \dots y_n$ in a FS form a basis:

- $y_1, \dots y_n$ are lin. independent;
- any solution y of the same DE is their lin. combination

$$y = C_1 y_1 + \dots + C_n y_n.$$

$\boxed{2e}$ Relation between two FS's.

We'll find such a relation for $n=2$. For $n>2$ everything will be similar.

Suppose $\{y_1, y_2\}$ and $\{\bar{y}_1, \bar{y}_2\}$ are two FSs. Since $\{y_1, y_2\}$ is a FS, then

some constants

$$\begin{aligned} \bar{y}_1 &= a_{11} y_1 + a_{12} y_2 \\ \bar{y}_2 &= a_{21} y_1 + a_{22} y_2 \end{aligned} \qquad (13)$$

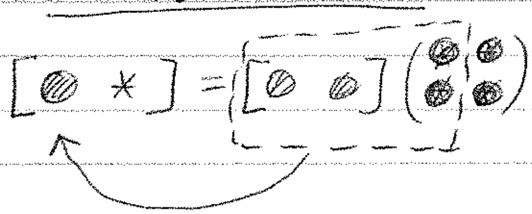
We can write this in a matrix form:

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \qquad (14)$$

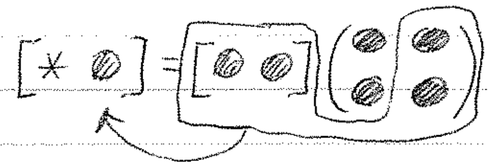
However, in order to make a connection to the theory to be developed in Chap. 4, we will write (13) using <u>row-vectors</u> $[y_1, y_2]$ and $[\bar{y}_1, \bar{y}_2]$ instead of the column-vectors $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$.

$$\underbrace{\bar{y}_1 = [y_1 \;\; y_2] \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}}, \qquad \underbrace{\bar{y}_2 = [y_1 \;\; y_2] \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}}$$

make this the 1st entry (column) of a new row-vector



make this the 2nd column entry



Put together:

$$[\bar{y}_1 \;\; \bar{y}_2] = [y_1 \;\; y_2] \underbrace{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}}_{A} \qquad (15a)$$

In general, for the n-th order linear DE (4):

$$[\bar{y}_1, \dots \bar{y}_n] = [y_1, \dots y_n] \cdot A \qquad (15b)$$

where $\{y_1, \dots y_n\}$ and $\{\bar{y}_1, \dots \bar{y}_n\}$ are two FS's and A is some constant matrix $(n \times n)$.

Eqs. (15) are stated as Thm. 3.9 in textook.

Corollary 1 (stated for $n=2$; similar for $n>2$). Let $\{y_1, y_2\}$ be a FS of a DE, and its Wronskian is $W(t)$. Let $\{\bar{y}_1, \bar{y}_2\}$ be another FS of same DE, and its Wronskian be $\overline{W}(t)$. Then, if

$\{y_1, y_2\}$ and $\{\bar{y}_1, \bar{y}_2\}$ are related by (15), their Wronskians are related by:

$$\boxed{\overline{W} = W \cdot \det(A)} \qquad (16)$$

<u>Proof</u> :

(15a) $\Rightarrow \qquad [\bar{y}_1 \;\; \bar{y}_2] = [y_1 \;\; y_2] \cdot A$

differentiate and use A = const matrix :

$\Rightarrow \qquad [\bar{y}_1' \;\; \bar{y}_2'] = [y_1' \;\; y_2'] \cdot A$

Put together as row 1 and row 2 :

$$\begin{bmatrix} \bar{y}_1 & \bar{y}_2 \\ \bar{y}_1' & \bar{y}_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} A \qquad (17)$$

Take the determinant of both sides and use a formula from Linear Algebra:

$$\det(P \cdot Q) = \det P \cdot \det Q$$

to get (16). ✓

<u>Corollary 2</u> If $\{y_1, \dots y_n\}$ and $\{\bar{y}_1, \dots \bar{y}_n\}$ are two FS's related by (15), then <u>matrix A is non-singular</u>.

<u>Proof</u> follows from:

- W of a FS $\neq 0$;
- $\det(A) \neq 0$ iff A is non-singular.

Ex. 3 (of formula (15) & Corollaries)

Consider DE

$$t^2 y''' + t y'' - y' = 0, \quad t > 0.$$

It is given that

$\{y_1, y_2, y_3\} = \{1, \ln t, t^2\}$ is its FS.

Determine if

$\{\bar{y}_1, \bar{y}_2, \bar{y}_3\} = \{2t^2 - 1, 3, t^2 - \ln t\}$ is a FS.

Sol'n:

1) Relate $\bar{y}_1$ etc to y_1, y_2, y_3 similarly to Eq. (13):

$$\bar{y}_1 = -1 \cdot y_1 + 0 \cdot y_2 + 2 \cdot y_3$$

$$\bar{y}_2 = 3 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3$$

$$\bar{y}_3 = 0 \cdot y_1 - 1 \cdot y_2 + 1 \cdot y_3$$

Following the steps that led from (13) to (15a), we obtain in our case:

$$[\bar{y}_1 \ \bar{y}_2 \ \bar{y}_3] = [y_1 \ y_2 \ y_3] \underbrace{\begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix}}_{A}.$$

2) By Corollary 2 (see also Eq. (16)), $\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ will be a FS if A is nonsingular, i.e. if $\det A \neq 0$.

expand over 2nd row

$$\det A = \begin{vmatrix} -1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} = (-1)\cdot(-1)^{2+3} \cdot \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \cdot (-1-6) = -7 \neq 0.$$

Thus, A is nonsingular $\Rightarrow \{\bar{Y}_1, \bar{Y}_2, \bar{Y}_3\}$ is a FS.

3 Nonhomogeneous linear DE

The n-th order DE (2) has the same superposition principle as its 2nd-order counterpart in Lecture 16:

$$\begin{pmatrix}\text{General sol'n} \\ \text{of a nonhomog.} \\ \text{linear DE}\end{pmatrix} = \begin{pmatrix}\text{particular} \\ \text{sol'n}\end{pmatrix} + \begin{pmatrix}\text{general sol'n} \\ \text{of homogen.} \\ \text{DE}\end{pmatrix}$$

$$y = y_p + y_c.$$

HW: Sec. 3.11.

1, 3 ← given $y_1, \dots y_n$: are they a FS? Solve IVP.
7, 8, 9 ← do sol's of a given IVP for a FS (from $W(t_0)$)?
11, 12, 13, 23, 25 ← Abel's Thm
16, 17, 18 ← FS of a given DE-2 satisfying given IC.
(add: verify that it is indeed a FS)
20, 21 ← topic $\boxed{2e}$: $\bar{Y} = YA$.

Sec. 3.13: # 23, 25 ← from given y_c & y_p, determine coefficients in DE and $g(t)$.