

Lecture 23General properties of linear systems of first-order DEs

We'll follow the outline of lecture 20, where we listed general properties of higher-order linear DEs for 1 variable.

In fact, the outline of topics 1-3 will mimic that of Lec. 20. But, before we get to topic 1, we'll establish a relation between higher-order DEs for 1 variable and systems of 1st order DEs.

① n<sup>th</sup> order scalar DEs as 1st-order linear systems

Ex. 1 Rewrite the 2nd-order DE

$$y'' - e^t \cdot y' + 3y = \sin 2t$$

as a 1st-order linear system.

Sol'n: Let  $y_1 = y$ ,  $y_2 = y'$ .

First eq. of the system is simply  $y_1' = y_2$   
( $y' = y'$ ).

Second eq. of the system will come from the given DE:

$$y'' = e^t \cdot y' - 3y + \sin 2t$$

$$\underbrace{(y_1')}' = e^t \cdot \underbrace{(y_1)'} - 3 \cdot \underbrace{y_1} + \sin 2t$$

$y_2 \qquad \qquad \qquad y_2 \qquad \qquad \qquad y_1$

$$y_2' = -3 \cdot y_1 + e^t \cdot y_2 + \sin 2t.$$

Putting them together:

$$y_1' = 0 \cdot y_1 + 1 \cdot y_2 \quad \Rightarrow$$

$$y_2' = -3 \cdot y_1 + e^t \cdot y_2 + \sin 2t$$

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\vec{y}'} = \underbrace{\begin{pmatrix} 0 & 1 \\ -3 & e^t \end{pmatrix}}_{\substack{P(t) \\ 2 \times 2 \text{ matrix}}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\vec{y}} + \underbrace{\begin{pmatrix} 0 \\ \sin 2t \end{pmatrix}}_{\vec{g}(t)}. \quad \underline{\underline{\quad}}$$

Ex. 2 Rewrite the 3rd-order DE

$$y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y = g(t)$$

as a 1st-order linear system.

Sol'n: Let  $y_1 = y$ ,  $y_2 = y'$ ,  $y_3 = y''$ .

First & second eqs. of the system are simply:

$$\begin{aligned} y_1' &= y_2 & \text{and} & & y_2' &= y_3 \\ (y' = y') & & & & (y')' &= y'' \end{aligned}$$

Third eq. of the system comes from the given DE:

$$\underline{y'''} = -p_2 \cdot \underline{y''} - p_1 \cdot \underline{y'} - p_0 \underline{y} + g(t)$$

$$\underbrace{(y'')}'_{y_3} = -p_0 \underbrace{y}_{y_1} - p_1 \underbrace{y'}_{y_2} - p_2 \underbrace{y''}_{y_3} + g(t)$$

$$y_3' = -p_0 y_1 - p_1 y_2 - p_2 y_3 + g(t).$$

Putting them together:

$$y_1' = 0 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3$$

$$y_2' = 0 \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 \quad \Rightarrow$$

$$y_3' = -p_0 y_1 - p_1 y_2 - p_2 y_3 + g(t).$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}.$$

So, same general form

$$\begin{array}{c} \vec{y}' \\ \uparrow \\ n \times 1 \text{ vector} \end{array} = \begin{array}{c} P(t) \\ \uparrow \\ n \times n \text{ matrix} \end{array} \begin{array}{c} \vec{y} \\ \uparrow \\ n \times 1 \text{-vector} \end{array} + \begin{array}{c} \vec{g}(t) \\ \uparrow \\ n \times 1 \text{-vector} \end{array} \quad (1)$$

Q: What do we achieve by rewriting a scalar higher-order DE as a lin. system?

A:

- unity of the approach to both these types of problems
- method of variation of parameters for nonhomogeneous DEs will become transparent
- unity of approach to numerical solution of both problems (this will not be considered in this course).

① Existence and uniqueness

Thm. 4.1 Consider an IVP that consists of the DE (system) (1) and an initial condition

$$\vec{y}(t_0) = \vec{y}_0 \quad (2)$$

Let the  $n^2$  coefficients  $p_{ij}(t)$  of matrix  $P(t)$  and  $n$  coefficients  $g_1(t), \dots, g_n(t)$  of vector  $\vec{g}$  be continuous on  $t \in (a, b)$ , and let  $t_0 \in (a, b)$ . Then the IVP has a unique sol'n on  $(a, b)$ .

In what follows we assume that conditions of this E&U thm. always hold.

② Properties of homogeneous linear systems

2a Principle of linear superposition

Thm. 4.2 Let  $\vec{y}_1, \dots, \vec{y}_r$  be solutions of the homogeneous linear DE system

$$\vec{y}' = P(t) \vec{y} \quad (3)$$

Then their linear combination

$$\vec{y} = c_1 \vec{y}_1 + \dots + c_r \vec{y}_r,$$

where  $c_1, \dots, c_r$  are any constants, is also a solution of (3).

Proof: book, p. 229 (same idea as before: Lec. 2, Lec. 20).

Note about notations: In topic ①, subscripts in  $y_1, y_2, \dots$  meant components of  $\vec{y}$ . Below, subscripts label different vectors  $\vec{y}_1, \vec{y}_2, \dots$

## 2b) Fundamental sets of sol'n's, and Wronskian

Let  $\vec{y}_1, \dots, \vec{y}_n$  be  $n$  solutions of the linear system (3) of  $n$  DEs.

Set  $\{\vec{y}_1, \dots, \vec{y}_n\}$  is called a FS of sol'n's if any sol'n of IVP (3)+(2) can be written as

$$\vec{y} = c_1 \vec{y}_1 + \dots + c_n \vec{y}_n \quad (4)$$

for some constants  $c_1, \dots, c_n$ .

(Somewhat conversely, if  $\{\vec{y}_1, \dots, \vec{y}_n\}$  is a FS, then (4) is called the general sol'n of (3).

Let us now determine a condition which determines whether a given set of sol'n's is a FS or not.

Suppose  $\{\vec{y}_1, \dots, \vec{y}_n\}$  is some set of sol'n's of (3)

Let's try to construct a sol'n of IVP (3)+(2) in the form (4). All we need is to satisfy the IC (2), because we know that (4) satisfies the DE (3). So:

$$c_1 \vec{y}_1(t_0) + c_2 \vec{y}_2(t_0) + \dots + c_n \vec{y}_n(t_0) = \vec{y}_0 \quad (5)$$

Using the VERY IMPORTANT PROPERTY (6) from Lec. 22 (p. 22-7):

$$[\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{A}_1 + x_2 \vec{A}_2 + \dots + x_n \vec{A}_n,$$

we rewrite (5) as:

$$\underbrace{[\vec{y}_1(t_0), \vec{y}_2(t_0), \dots, \vec{y}_n(t_0)]}_{\text{let's call this matrix}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\vec{c} \leftarrow n \times 1 \text{ vector}} = \vec{y}_0 \quad (6a)$$

"Psi"  
 $\vec{\Psi}(t_0)$

Then, (6a) can be rewritten as

$$\vec{\Psi}(t_0) \vec{c} = \vec{y}_0. \quad (6b)$$

Then:

- We know that the sol'n of IVP (3)+(2) exists by the E & U theorem (topic ①).
- This is equivalent to the fact that the constants  $c_1, \dots, c_n$  in (5), or vector  $\vec{c}$  in (6b), can be found for any  $\vec{y}_0$ .
- From Lin. algebra we know that this can be guaranteed by  $\vec{\Psi}(t_0)$  being nonsingular, or

$$W(t) \equiv \boxed{\det \vec{\Psi}(t_0) \neq 0} \quad (7)$$

↑  
Wronskian

↑ condition for  
 $\{\vec{y}_1, \dots, \vec{y}_n\}$  to form a FS.

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Ex. 3(a) Verify that  $\vec{y}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$ ,  $\vec{y}_2 = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$   
 form a FS of sol'ns for

$$\vec{y}' = \begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix} \vec{y}, \quad t > 0.$$

Sol'n: 1) We first need to check that  $\vec{y}_1, \vec{y}_2$  are solutions. Do this for  $\vec{y}_1$ ; for  $\vec{y}_2$  it is similar.

lhs:  $\vec{y}_1' = \begin{pmatrix} t \\ 1 \end{pmatrix}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

rhs:  $\begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + 1 \\ t \cdot t^{-2} - t^{-1} \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

lhs = rhs,  $\checkmark$ .

2) To verify if they form a FS, we need to find their Wronskian at some point  $t > 0$ . The easiest point is  $t_0 = 1$ :

$$W(t_0=1) = \begin{vmatrix} 1 & 1^{-1} \\ 1 & -1^{-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0,$$

$\Rightarrow \{ \vec{y}_1, \vec{y}_2 \}$  form a FS.

Note: This lin. system is just a 2nd-order DE in disguise:

$$y_1' = 0 \cdot y_1 + 1 \cdot y_2 \Rightarrow y_1' = y_2$$

$$y_2' = \frac{1}{t^2} y_1 - \frac{1}{t} y_2 \quad (y_1')' = \frac{1}{t^2} y_1 - \frac{1}{t} (y_1)'$$

$$\Rightarrow y_1'' + \frac{1}{t} y_1' - \frac{1}{t^2} y_1 = 0.$$

(Compare with Ex. 1 & Ex. 2.)

Ex. 3(b) Find the solution of that lin. system satisfying the initial cond.

$$\vec{y}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Sol'n: We follow the steps by which we obtained the Wronskian of a lin. system (see Eq. (7)).

$$\text{General sol'n: } \vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 = [\vec{y}_1, \vec{y}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\vec{y}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow [\vec{y}_1(1), \vec{y}_2(1)] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

We exhibited this matrix  $\uparrow$  in Ex. 3(a),  $\Rightarrow$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \underset{\uparrow}{=} \frac{1}{-2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Note: It is a coincidence  $\rightarrow$  [formula for a  $2 \times 2$   $A^{-1}$  on p. 217] that  $A^{-1}$  and  $A$  look proportional.

In general they are not! Again, see p. 217.

$$= -\frac{1}{2} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 1/2 \end{pmatrix}.$$

$$\text{Thus, } \vec{y}(t) = -\frac{5}{2} \vec{y}_1(t) + \frac{1}{2} \vec{y}_2(t)$$

$$= -\frac{5}{2} \begin{pmatrix} t \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}.$$



We will now state a counterpart of the Abel Thm. for  $W(t)$  of a scalar DE:  
 $W'(t) = -p_{n-1}(t) W(t)$ .

For lin. system, this is called Liouville's formula

First, we need a definition.

Def: Let  $P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix}$ .

Then trace of P is:

$tr [P] = p_{11} + p_{22} + p_{33} + \dots + p_{nn}$  (sum of diagonal elements).

Thm. 4.4 (Liouville's formula).

Let  $\{\vec{y}_1, \dots, \vec{y}_n\}$  be a set of sol'ns of lin. system (3) ( $\vec{y}' = P\vec{y}$ ) and let  $W$  be the Wronskian of these solutions. Then

$W'(t) = tr [P] \cdot W(t)$ . (8a)

Solving (8a), we get:

$W(t) = W(t_0) \cdot e^{\int_{t_0}^t tr [P(s)] ds}$ . (8b)

Thus, as before, the Wronskian is either 0 or nonzero for all  $t$  in the interval of existence and uniqueness of the solution.

Note: For scalar DEs of order  $n$ , Liouville's formula reduces to Abel's Thm.  
 E.g., in Ex. 2 we showed that

$$y''' + p_2 y'' + p_1 y' + p_0 y = 0 \quad (\text{DE-3})$$

is equivalent to

$$\vec{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix} \vec{y}, \quad \vec{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}. \quad (\text{lin. sys.})$$

For (DE-3), Abel's Thm. yields

$$W'(t) = -p_2(t)W.$$

For (lin. sys.),  $\text{tr}[P] = 0 + 0 + (-p_2)$ ,  $\Rightarrow$

Liouville's formula yields

$$W'(t) = (-p_2)W,$$

which is indeed the same.

Proof of Liouville's formula for  $n=2$  is in ECH.

2c FS of solutions of lin. systems always exist.  
 (a.k.a. how to find a FS).

The idea is the same as for scalar DEs.

Consider  $n$  IVPs which share the same lin. system (3):

$$\vec{y}' = P \vec{y} \quad (\text{for any } \vec{y} \text{ from } \{\vec{y}_1, \dots, \vec{y}_n\})$$

but different initial conditions, which we illustrate for  $n=3$ :

$$\vec{y}_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{y}_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{y}_3(t_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Each of these solutions exist by the E&U thm. Moreover, their Wronskian at  $t_0$  is:

$$W(t_0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

$\Rightarrow$  by Liouville's formula  $W(t) \neq 0$  in the interval of E&U.

### 2d Solutions in a FS are lin. independent

If  $\{\vec{y}_1, \dots, \vec{y}_n\}$  is a FS, then the fund.

matrix  $\Psi(t) = [\vec{y}_1, \dots, \vec{y}_n]$  is nonsingular

because  $\det \Psi = W \neq 0$ . But it is

known from lin. algebra that in any nonsingular matrix, columns ( $\vec{y}_1, \dots, \vec{y}_n$  in this case) are lin. independent.

### 2e Relation between two FSs

Let  $\Psi = [\vec{y}_1, \dots, \vec{y}_n]$  and  $\hat{\Psi} = [\hat{\vec{y}}_1, \dots, \hat{\vec{y}}_n]$  be

fundamental matrices of the same  
lin. system (3). Then:

Thm. 4.5 (a) There exists a unique matrix  $C$   
s.t.  $\hat{\Psi} = \Psi C$ . (9)

(b) and  $C$  is nonsingular.

Proof (for  $n=2$ )

(a) Since  $\{\vec{y}_1, \vec{y}_2\}$  is a FS,  $\Rightarrow$  by the linear  
Superposition  
Principle  
(topic 2 [a])

$$\hat{\vec{y}}_1 = \underbrace{C_{11}}_{\text{some constants}} \vec{y}_1 + \underbrace{C_{21}}_{\text{some constants}} \vec{y}_2 = [\vec{y}_1, \vec{y}_2] \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \Psi \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}.$$

Similarly,  $\hat{\vec{y}}_2 = C_{12} \vec{y}_1 + C_{22} \vec{y}_2 = \Psi \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix}.$

Thus,  $[\hat{\vec{y}}_1, \hat{\vec{y}}_2] = [\Psi \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}, \Psi \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix}]$

$$\hat{\Psi} = \Psi C. \quad \checkmark$$

by Fact 4 on p. 22-6

$$(b) \quad \underbrace{\det \hat{\Psi}}_{\neq 0} = \det(\Psi C) = \underbrace{\det \Psi}_{\neq 0} \cdot \det C$$

$\Rightarrow \det C \neq 0 \Rightarrow C$  is nonsingular.

How to find this  $C$ ?

$$\hat{\Psi} = \Psi \cdot C \Rightarrow \Psi^{-1} \hat{\Psi} = \overbrace{\Psi^{-1} \Psi}^I \cdot C \Rightarrow \boxed{\Psi^{-1} \hat{\Psi} = C}$$

③ Nonhomogeneous linear systems of DEs

$$\vec{y}' = P\vec{y} + \vec{g} \quad (1)$$

As before:

$$\left( \begin{array}{c} \text{General} \\ \text{sol'n of} \\ \text{nonhom. system} \end{array} \right) = \left( \begin{array}{c} \text{General} \\ \text{sol'n of} \\ \text{homog. system} \end{array} \right) + \left( \begin{array}{c} \text{Particular} \\ \text{sol'n of} \\ \text{nonhom. system} \end{array} \right)$$

$$\vec{y} = \vec{y}_c + \vec{y}_p$$

We also have a superposition principle for nonhomogeneous systems:

Thm. 4.8 (Sec. 4.8). Let  $\vec{v}_1$  solve (1) with  $\vec{g} = \vec{g}_1$  and  $\vec{v}_2$  solve (1) with  $\vec{g} = \vec{g}_2$ . Then  $(a_1\vec{v}_1 + a_2\vec{v}_2)$  solves (1) with  $\vec{g} = a_1\vec{g}_1 + a_2\vec{g}_2$ , where  $a_1, a_2$  are any constants.

MW, Sec. 4.2

- # 7 ← rewrite as  $\vec{y}' = A\vec{y}$ .
- # 11, 13 ← scalar DE to lin. sys.
- 15, 16 ← lin. sys. as scalar DE
- 19, 21 ← 2 DE-2s as lin. sys.
- # 9 ← rewrite as  $\vec{y}' = A\vec{y}$ , verify sol'n; solve IVP.
- # 1, 3 ← rewrite as  $\vec{y}' = P\vec{y}$ , verify sol'n
- # 7, 10 ← verify if a FS
- # 15, 17, 23 ← verify if FS, solve IVP.
- # 25, 26, 28 ← Liouville's Thm.
- # 29, 30, 31 ← all (b,c) ←  $\hat{\psi} = \psi C$ ; # 33 ← similar.

Sec. 4.3

This is from Sec. 4.2

Sec. 4.8. #11 given  $P, \vec{y}$ , find  $\vec{g}$ .