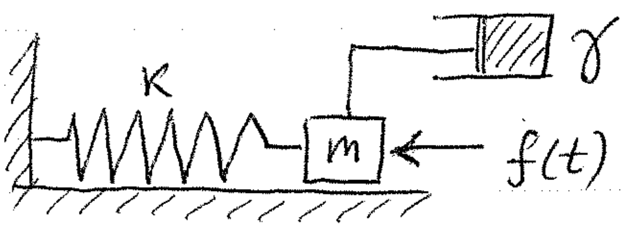
Lecture 26 Laplace transform (LT): Motivation and Introduction.

Although LT can be developed to systems of linear differential eqs, we'll consider it only for the scalar, second-order DEs.

A necessary condition to be able to apply LT is that the DE had constant coefficients.

(i) Motivation



- Suppose we have a vibrating system whose model we know to be

$$m y'' + \gamma y' + k y = f(t), \qquad (1)$$

but we do not know m, γ, k (say, the system is in a black box).

- Suppose we drive the system with a known force $f(t)$ and measure its response $y(t)$.
- From this response, the form $f(t)$, and the known form of (1) (oscillator with damping), can we compute the parameters

m, γ, k of the system?

Answer 1: It is possible, but very tedious and technically complex to do if we use solution techniques developed so far (in Lec. 15 & 19).

Answer 2: It is very easy if we use the method of Laplace transform.

So the main usefulness of LT is that it allows one to deduce the system properties by measuring its response to a known input (force).

(2) "Bird's view" idea of LT

Instead of working with functions in the time domain (i.e., $y(t)$ or $f(t)$ or $g(t)$), LT works in a transform domain described by some variable s.

So, Laplace transform does this:

$$f(t) \xrightarrow{LT} F(s)$$

(we will illustrate details shortly).

Thus, instead of solving an IVP in the time domain:

$$\boxed{\begin{array}{c}\text{IVP} \\ \text{with input } f(t)\end{array}} \xrightarrow[\substack{\text{domain,} \\ \text{complicated} \\ \text{solution}}]{\text{time}} \boxed{\text{Solution } y(t)}$$

We go in a round-about way:

$$\begin{tikzcd} \boxed{\text{IVP with } f(t)} \arrow[d, "LT"] & & \boxed{y(t)} \\ \boxed{F(s)} \arrow[rr, "\text{Easy solution in s-domain:}"] & & \boxed{Y(s)} \arrow[u, "\text{inverse LT}"'] \end{tikzcd}$$

$$Y(s) = F(s)\cdot \underbrace{\Phi(s)}_{\text{some function depending on the IVP}}$$

Thus, we need to learn 3 steps:

1) How to find LT of $f(t)$;
2) How to solve the IVP in the s-domain; (in particular, how to find $\Phi(s)$);
3) How to transform the "solution" $Y(s)$ from the s-domain back to the t-domain.

3 Definition of LT

<u>Def.</u> Let $f(t)$ be a function defined on $t \in [0, \infty)$. Then its LT:

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)\, e^{-st}\, dt, \qquad (2)$$

provided that this <u>improper integral</u> exists (Calc. 2)

<u>Note 1</u>: Unlike in Lec. 25, where "s" was just another notation for the time variable, here "s" is a <u>completely different</u> variable <u>not related to time at all</u>!

<u>Note 2</u> Recall from Calc 2:

$$\int_0^\infty f(t)e^{-st}\,dt = \lim_{T\to\infty} \int_0^T f(t)e^{-st}\,dt \qquad (3)$$

(If the integral exists, it is said to <u>converge</u>, otherwise it is said to diverge.)

<u>Ex. 1</u> Find the LT, if it exists, of:

(a) $f(t)=e^{at}$; (b) $f(t)=t$; (c) $f(t)=e^{t^2}$.

<u>Sol'n:</u> (a) $F(s)=\int_0^\infty e^{at}\cdot e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$

$$= \lim_{T\to\infty}\int_0^T e^{(a-s)t}\,dt = \lim_{T\to\infty}\frac{e^{(a-s)T}-1}{a-s} = \begin{cases}\infty, & a>s\\ \frac{1}{s-a}, & a<s.\end{cases}$$

Thus, $\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$, $s>a$.

(b) $F(s)=\int_0^\infty te^{-st}\,dt = \lim_{T\to\infty}\int_0^T te^{-st}\,dt$

int. by parts (Calc. 2)

$$\overset{\vee}{=} \lim_{T\to\infty}\begin{cases} T^2/2, & s=0\\ \frac{1}{s^2}\Big(\underbrace{-e^{-sT}(1+sT)}_{\text{this}\to 0 \text{ as } T\to\infty}+1\Big), & s>0\end{cases}$$

because e^{-sT} "wins" over T.

$$= \frac{1}{s^2}(0+1) = \frac{1}{s^2},\quad s>0.$$

Thus, $\mathcal{L}\{t\}=\frac{1}{s^2}$, $s>0$.

(c) $F(s) = \int_0^\infty e^{t^2} \cdot e^{-st}\, dt$

$$= \lim_{T\to\infty} \int_0^T e^{t^2 - st}\, dt.$$

This integral cannot be computed as an "elementary" function (i.e. that studied in Calculus). However, as $t \to T \to \infty$,

$t^2 > st$ for any <u>given</u> value of s.

Therefore, $e^{t^2 - st} \approx e^{(\to +\infty)} \to \infty$,

and so the integral in this case <u>diverges</u> (does not exist).

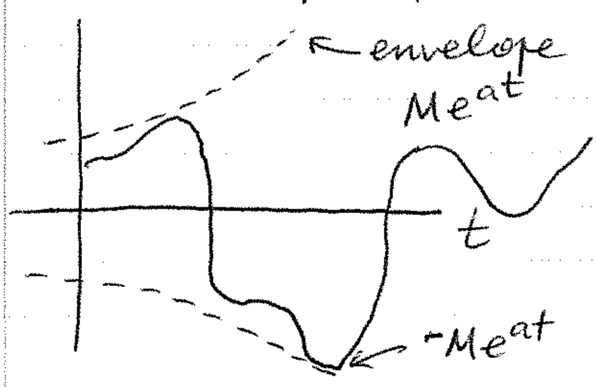
Thus, $\mathcal{L}\{e^{t^2}\}$ does not exist.

4 <u>Existence of LT</u>

Ex. **1**(c) motivates the following definition:

<u>Def:</u> Function $f(t)$ defined <u>on $t \in [0,\infty)$</u> is called <u>exponentially bounded</u> if there are <u>constants</u> $M \geq 0$ and a s.t.

$$|f(t)| \leq M e^{at} \text{ for all } t > 0.$$



Basially, this means that for $t \to \infty$, we do <u>not</u> want $f(t)$ to <u>grow</u> faster than e^{at} for some a.

In Ex. 1(c), $f(t)=e^{t^2}$ is not exponentially bounded, as it grows faster than e^{at} for any given a.

Thm. 5.1 (Existence of LT).

Let $f(t)$ be piecewise continuous and exponentially bounded on $t \in [0,\infty)$. Then its LT

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

exists for all $s>a$.
(Examples: Ex. 1(a); 1(b) $(a=0)$.)

In what follows we will always assume that $f(t)$ is such that its LT exists.

5 Linear superposition principle for LT

Thm. 5.2 Let $f_1(t)$ & $f_2(t)$ be such that their LT exist for $s>a$ (for some a). Let c_1, c_2 be arbitrary constants. Then

$$\mathcal{L}\{c_1 f_1(t)+c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\}.$$

6 The Inverse Laplace Transform and the issue of Uniqueness

Thm. 5.3 Suppose that $f_1(t)$, $f_2(t)$ are continuous on $[0,\infty)$. Let $F_1(s)$, $F_2(s)$ be their respective LT, which we assume exist for $s>a$. Then

$$\begin{pmatrix} f_1(t) = f_2(t) \\ \text{for all } t > 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} F_1(s) = F_2(s) \\ \text{for all } s > a \end{pmatrix}.$$

That is, if LT's of two continuous function are equal, so are the original functions.

<u>Note</u>: If f_1, f_2 are only piecewise continuous, then they are equal at all points of their continuity (see book, p. 325).

This Thm. motivates a method of finding the <u>inverse</u> of LT: build a <u>look-up table</u> of LT's:

function	LT
$f_1(t)$ $\rightleftarrows$	$F_1(s)$
$f_2(t)$ $\rightleftarrows$	$F_2(s)$
$\vdots$	$\vdots$

Since by Thm. 5.3 there is a <u>unique</u> correspondence between $f(t)$ and its $F(s)$, then from the table we can find $f(t)$ given a $F(s)$.

So far we have, from Ex. 1(a), (b)

function	LT
e^{at}	$\frac{1}{s-a}$, $s > a$
t	$\frac{1}{s^2}$, $s > 0$

Let us show how it can be used.

Ex. 2 Let $F(s) = \frac{2s}{s^2-1}$, $s > 1$.

What is $f(t)$? (I.e., find the inverse LT of $\frac{2s}{s^2-1}$.)

Sol'n: The given $F(s)$ is not in our table. However, we can use Partial Fraction Expansion (review it from Calc. 2!) to write:

$$\frac{2s}{s^2-1} = \frac{1}{s-1} + \frac{1}{s+1}$$

Again, Review the method of Part. Fractions on how to find these coefficients.

Now, both $\frac{1}{s-1}$ & $\frac{1}{s+1}$ are in our table: it is $\frac{1}{s-a}$ with $a=1$ & $a=-1$.

Then, denoting the inverse LT by $\mathcal{L}^{-1}$ and using the linear superposition principle (Thm. 5.2), we have:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

from table $\longrightarrow$ $= e^{t} + e^{-t}$.

Thus, $\mathcal{L}^{-1}\left\{\frac{2s}{s^2-1}\right\} = e^{t} + e^{-t}$.

<u>HW</u>: Sec. 5.1.

1, 2, 3, 4 ← find a LT of a simple function

9 ← shifted 1 (Hint: use $t=(t-1)+1$ in the exponent and use a new variable $z=t-1$ instead of t.)

13, 15 ← LT of t^n via int. by parts (tell them that $\lim_{T\to\infty} T^n e^{-sT}=0\ \forall s>0$).

16, 17 ← LT of $\sin\omega t$, $\cos\omega t$

18, 19 ← shifted of the same. Hint: Similar to #9: $t=(t-2)+2$.

32, 33, 35 ← improper $\int$ (just a Calc. 2 exercise)

23 ← linearity of LT

37, 39 ← similar to my Ex. 2