Lecture 8 An application of nonlinear DEs: the Logistic model

① Population with limited resources

For a population with unlimited resources (Lecture 4), we had

$$\frac{dP}{dt} = r_b P - r_d P$$

birth rate was independent of P.

However, when members of the population begin competing for food or living space, one can write:

$$r_b = r_0 (1 - kP)$$

some const > 0.

Then

$$\frac{dP}{dt} = \underbrace{r_0(1-kP)}_{r_b} P - r_d P$$

$$= (r_0 - r_d) P - r_0 k P^2 \equiv aP - bP^2.$$

some positive constants.

$$= a\left(1 - \frac{b}{a} P\right) P.$$

In notations of the textbook:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{P_e}\right) P \qquad (1)$$

Note: Equilibrium solutions are for $dP/dt = 0$

$\Rightarrow$ $P = 0$ or $1 - P/P_e = 0 \Rightarrow P = P_e$ "equilibrium".

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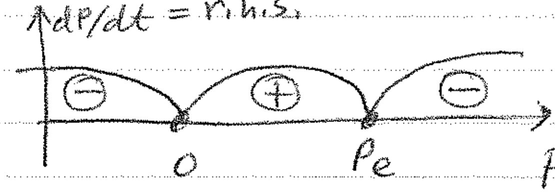
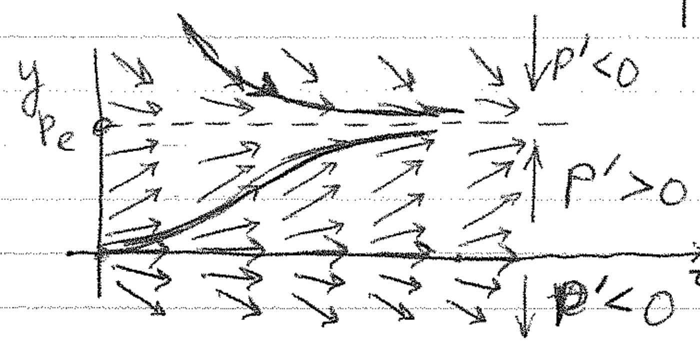
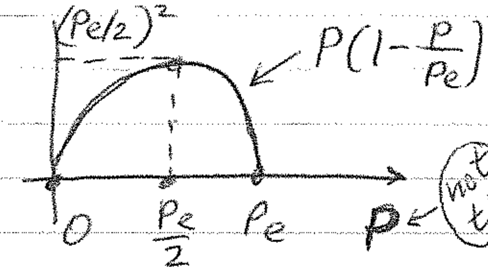
(2) Qualitative behavior of solution

$$\frac{dP}{dt} = r\left(1-\frac{P}{P_e}\right)P$$







So, as $t \to \infty$, $P(t) \to P_e$ for any $P(0) > 0$.

(3) Exact analytical solution.

You solved the DE $\frac{dP}{dt} = P(a-P)$ at least twice in HW5/Sec. 2.6 and then also, by another method, in HW7/Sec. 2.5.

Also, the solution is presented in detail on pp. 71-72. $\boxed{\text{MUST READ IT.}}$

Result:

$$P(t) = \frac{P(0)\cdot P_e}{P(0) - (P(0)-P_e)e^{-rt}} \qquad (2)$$

Let us verify the behavior for $t \to \infty$:

$e^{-rt} \to 0 \quad \Rightarrow$

$$P(t) \to \frac{P(0)\,P_e}{P(0) - 0} = P_e. \quad \checkmark$$

as predicted earlier

④ Constant migration

$$P' = r(1-P)P + \underbrace{M}_{\text{constant migration.}}$$

$\underline{M>0}$ — people move $\underline{\text{in}}$
$\underline{M<0}$ — people move $\underline{\text{out}}$

<u>Note:</u> A more general form would be

$$P' = r\left(1-\frac{P}{P_*}\right)P + M$$

some const., but <u>not</u> P_e any more!

but we will set $P_* = 1$ for simplicity.

<u>Ex. 1</u> Model <u>with</u> migration can be <u>transformed</u> into one without, and hence analyzed.

Find the <u>qualitative</u> behavior of the solution of:

$$P' = (1-P)P \underbrace{- \tfrac{2}{9}}_{M<0 \Rightarrow \text{emigration}}$$

<u>Sol'n:</u>

0) <u>Preliminaries about a linear model.</u>

Recall $y' = ay + b$ (const) from Lecture 2.

$$y' = a(y + \underbrace{\tfrac{b}{a}}_{-c}) \quad \Rightarrow$$

(3)

$$y' = a(y-c)$$

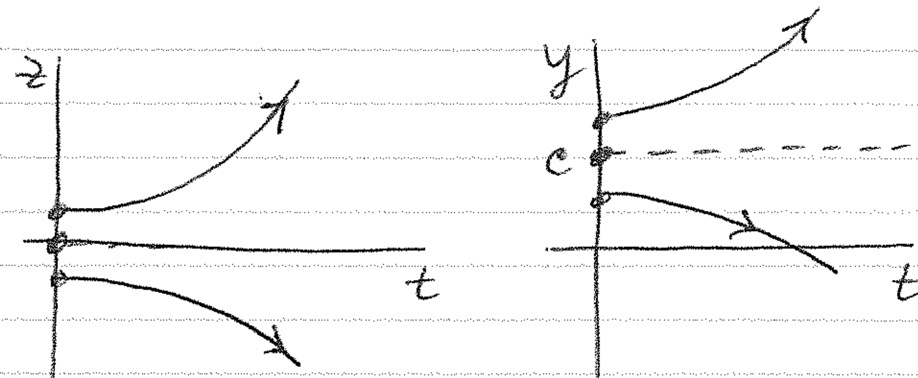
since $c'=0$

$$\underbrace{(y-c)}_{z}' = a\underbrace{(y-c)}_{z}$$

$$z' = az.$$

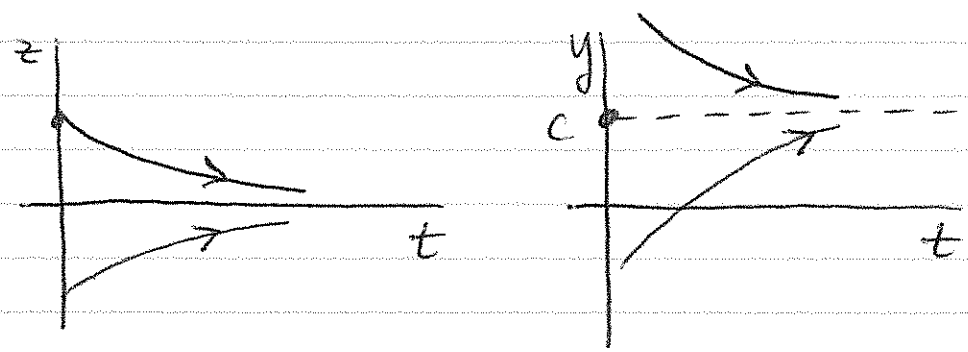
$a>0$

$z = Ce^{at}$



(The only) Equilibrium $y=c$ is unstable (solution moves away from it).

$a<0$



(the only) Equilibrium $y=c$ is stable (all solutions move towards it).

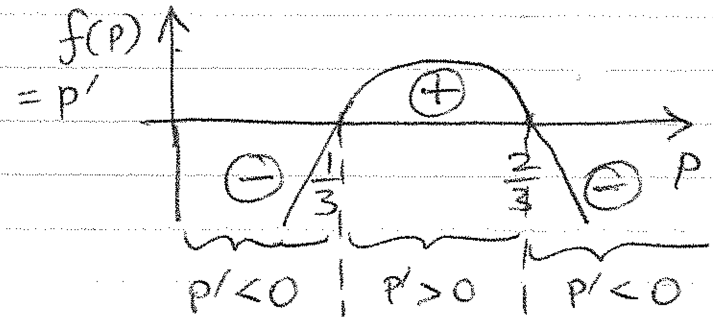
1) Come back to our nonlinear problem.

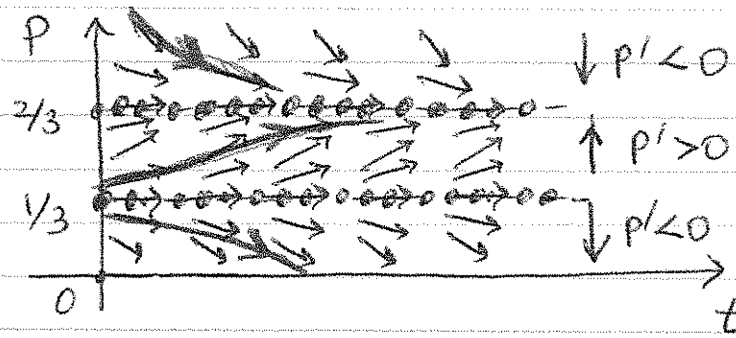
$$P' = (1-P)P - \tfrac{2}{9} = P - P^2 - \tfrac{2}{9} \equiv \underbrace{f(P)}_{\text{quadratic function}}$$

$$= -(P-\tfrac{1}{3})(P-\tfrac{2}{3})$$

roots of $f(P)$

Method 1 of analysis: look at (sign of $f(P)$) = (sign of P').





We find:

- there are two equilibria, $(P_e)_1 = \frac{1}{3}$, $(P_e)_2 = \frac{2}{3}$. (Multiple equilibria can occur only in nonlinear models, but not in a linear one.)
- $(P_e)_1 = \frac{1}{3}$ is the unstable equilibrium (solutions move away from it)
- $(P_e)_2 = \frac{2}{3}$ is the stable equilibrium (solutions that are sufficiently close to it ($\frac{1}{3} < P(0) < \infty$) move towards it).

Existence of stable and unstable equilibria is common for nonlinear models.

Method 2 of analysis:
Consider behavior close to $(Pe)_1$ & $(Pe)_2$

$$P' = -\left(P-\tfrac{1}{3}\right)\left(P-\tfrac{2}{3}\right).$$

a) Let $P \approx (Pe)_2 = \frac{2}{3}$

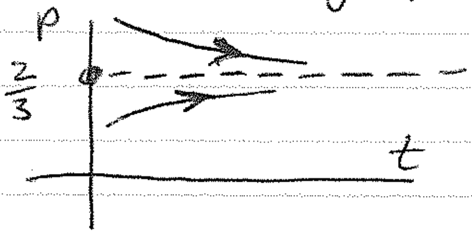
$$P' = -\Big(\underbrace{\underbrace{\approx\tfrac{2}{3}}_{P} - \tfrac{1}{3}}\Big)\cdot\left(P-\tfrac{2}{3}\right)$$

$\ominus \cdot \oplus\ \left(=\tfrac{2}{3}-\tfrac{1}{3}=\tfrac{1}{3}\right)$, so $\left(-1\cdot\tfrac{1}{3}\right)$

So, near $(Pe)_2 = \frac{2}{3}$, $\quad P' \approx -\frac{1}{3}\cdot\left(P-\frac{2}{3}\right)$

This is Eq. (3) with $a \approx -\frac{1}{3} < 0$,

and then by p. 8-4, $Pe = \frac{2}{3}$ is stable.



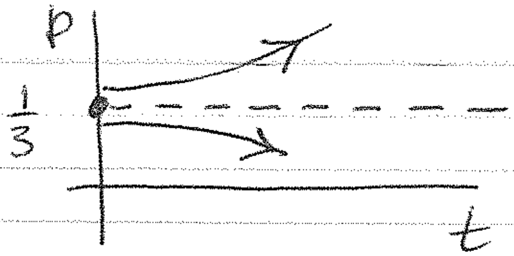
} all solutions that are sufficiently close to $P_e = 2/3$ tend towards it for $t \to \infty$.

b) Let $P \approx (Pe)_1 = \frac{1}{3}$

$$P' = -\Big(\underbrace{\underbrace{P}_{} - \tfrac{2}{3}}_{\approx -\frac{1}{3}}\Big)\left(P-\tfrac{1}{3}\right)$$

$\ominus \quad \approx\frac{1}{3}-\frac{2}{3} \approx \left(-\frac{1}{3}\right)$, so $-1\cdot\left(-\frac{1}{3}\right) = \frac{1}{3}$.

So, near $(Pe)_1 = 1/3$, $\quad P' \approx \left(+\frac{1}{3}\right)\cdot\left(P-\frac{1}{3}\right)$.
This is Eq. (3) with $a = +\frac{1}{3} > 0$, and then by p. 8-4, $Pe = 1/3$ is unstable.



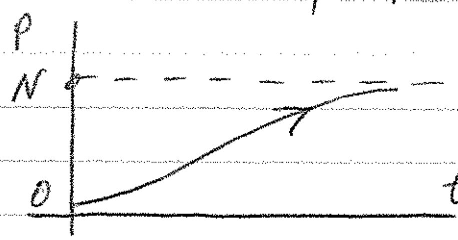
all solutions that are sufficiently close to $P_e = 1/3$ tend **away** from it as $t \to \infty$.

(5) Other applications of the logistic model

There are many.

a) Infectious disease w/o recovery (p. 74)

$$\frac{dP}{dt} = k \cdot \underset{\uparrow\ \text{\# of infected}}{P} \cdot \underbrace{(\overset{\text{total}}{N} - P)}_{\text{\# of non-infected}}$$



Everyone will get infected eventually.

b) Quadratic drag force (pp. 79-83) ← optional.

HW. Sec. 2.8: 1, 2, 3; 4, 5, 6, 7; 18, 19.

Hint for #18: See Eq. (7) in Sec. 2.8 and use the exact solution of Eq. (1) (also in the book).

Answers:
- #2: $10 \ln 4.4$
- #4: $P_e = 1/4, 3/4$; $P_e = 3/4$ is stable
- #6: $P_e = 1/2$; $P(t) \to -\infty$