Solution In an analytic geometry course, we would probably find the equation of the line by first calculating the slope of the line. In this example, however, we are interested in developing methods that can be used to find equations for more complicated curves; and we do not want to use special purpose techniques, such as slopes, that apply only to lines.

Since the points (1, 2) and (3, 7) lie on the line defined by $dx + ey + f = 0$, we insert these values into the equation and find the following conditions on the coefficients d, e, and f:

$$\begin{aligned} d + 2e + f &= 0 \\ 3d + 7e + f &= 0. \end{aligned}$$

We are guaranteed from Theorem 4 that the preceding homogeneous linear system has nontrivial solutions; that is, we can find a line passing through the two given points. To find the equation of the line, we need to solve the system. We begin by forming the associated augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 7 & 1 & 0 \end{bmatrix}.$$

The preceding matrix can be transformed to reduced echelon form, yielding

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}.$$

It follows that the solution is $d = -5f$, $e = 2f$, and hence the equation of the line is

$$-5fx + 2fy + f = 0.$$

Canceling the parameter f, we obtain an equation for the line:

$$-5x + 2y + 1 = 0.$$

Example 8 suggests how we might determine the equation of a conic that passes through a given set of points in the xy-plane. In particular, see Eq. (3); the general conic has six coefficients, $a, b, \ldots, f$. So, given any five points (x_i, y_i) we can insert these five points into Eq. (3) and the result will be a homogeneous system of five equations for the six unknown coefficients that define the conic section. By Theorem 4, the resulting system is guaranteed to have a nontrivial solution—that is, we can guarantee that any five points in the plane lie on the graph of an equation of the form (3). Example 9 illustrates this point.

EXAMPLE 9 Find the equation of the conic section passing through the five points $(-1, 0)$, $(0, 1)$, $(2, 2)$, $(2, -1)$, $(0, -3)$. Display the graph of the conic.

Solution The augmented matrix for the corresponding homogeneous system of five equations in six unknowns is listed below. In creating the augmented matrix, we formed the rows in the same order the points were listed and formed columns using the same order the unknowns were listed in Eq. (3). For example, the third row of the augmented matrix arises from inserting (2, 2) into Eq. (3):

$$4a + 4b + 4c + 2d + 2e + f = 0.$$

In particular, the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 4 & 4 & 4 & 2 & 2 & 1 & 0 \\ 4 & -2 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 9 & 0 & -3 & 1 & 0 \end{bmatrix}.$$

We used MATLAB to transform the augmented matrix to reduced echelon form, finding

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 7/18 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -11/18 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2/3 & 0 \end{bmatrix}.$$

Thus, the coefficients of the conic through these five points are given by

$$a = -7f/18, b = f/2, c = -f/3, d = 11f/18, e = -2f/3.$$

Setting $f = 18$, we obtain a version of Eq. (3) with integer coefficients:

$$-7x^2 + 9xy - 6y^2 + 11x - 12y + 18 = 0.$$

The graph of this equation is an ellipse and is shown in Fig. 1.7. The graph was drawn using the contour command from MATLAB. Contour plots and other features of MATLAB graphics are described in the Appendix.

Finally, it should be noted that the ideas discussed above are not limited to the xy-plane. For example, consider the quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0. \tag{4}$$

The graph of Eq. (4) is a surface in three-space; the surface is known as a ***quadric surface***. Counting the coefficients in Eq. (4), we find ten. Thus, given any nine points in three-space, we can find a quadric surface passing through the nine points (see Exercises 30–31).

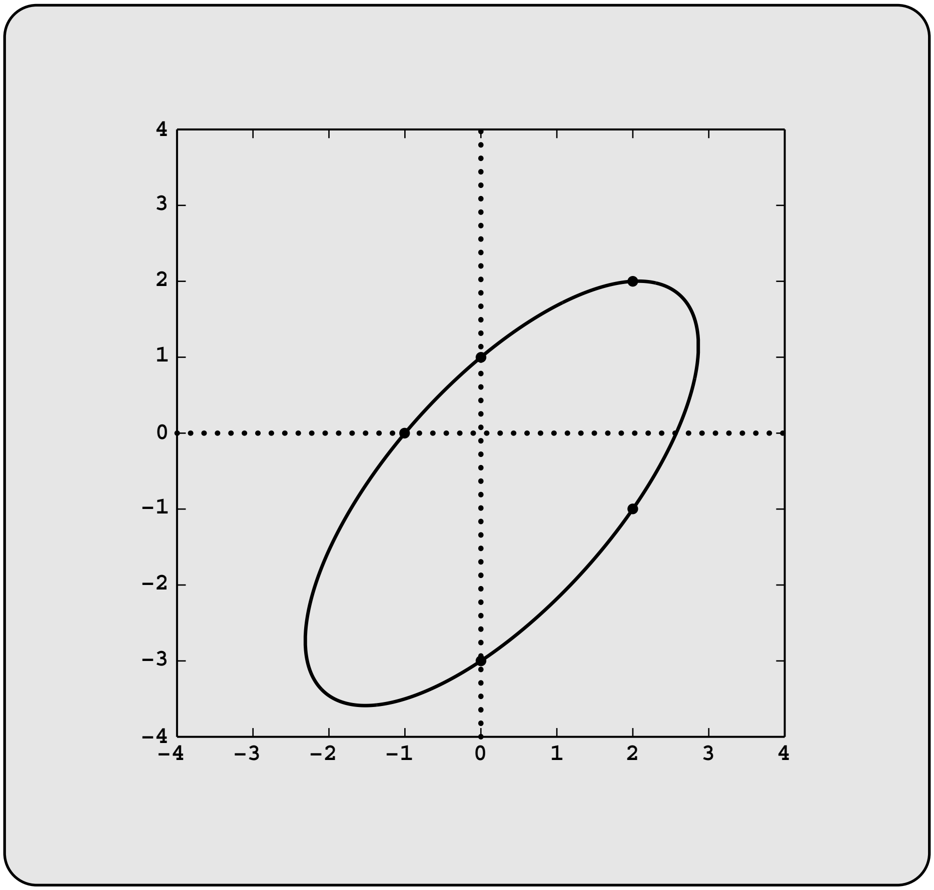

Figure 1.7 The ellipse determined by five data points, see Example 9.

1.3 EXERCISES

In Exercises 1–4, transform the augmented matrix for the given system to reduced echelon form and, in the notation of Theorem 3, determine n, r, and the number, $n - r$, of independent variables. If $n - r > 0$, then identify $n - r$ independent variables.

1.
$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= 1 \\ -2x_1 - 2x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 + 5x_3 &= 5 \\ -2x_1 - 2x_2 - 2x_3 &= -3 \end{aligned}$$

2.
$$\begin{aligned} 2x_1 + 2x_2 &= 1 \\ 4x_1 + 5x_2 &= 4 \\ 4x_1 + 2x_2 &= -2 \end{aligned}$$

3.
$$\begin{aligned} -x_2 + x_3 + x_4 &= 2 \\ x_1 + 2x_2 + 2x_3 - x_4 &= 3 \\ x_1 + 3x_2 + x_3 \phantom{{}+x_4} &= 2 \end{aligned}$$

4.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 5x_4 &= 2 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 1 \\ -x_1 - 2x_2 - 3x_3 + 7x_4 &= 2 \end{aligned}$$

In Exercises 5 and 6, assume that the given system is consistent. For each system determine, in the notation of Theorem 3, all possibilities for the number, r of nonzero rows and the number, $n - r$, of unconstrained variables. Can the system have a unique solution?