The idea of the vector form for the general solution is straightforward and is best explained by a few examples.

Example 2 The matrix B is the augmented matrix for a homogeneous system of linear equations. Find the general solution for the linear system and express the general solution in terms of vectors

$$B = \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Solution Since B is in reduced echelon form, it is easy to write the general solution:

$$x_1 = x_3 + 3x_4, \qquad x_2 = -2x_3 - x_4.$$

In vector form, therefore, the general solution can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

This last expression is called the *vector form for the general solution*.

In general, the vector form for the general solution of a homogeneous system consists of a sum of well-determined vectors multiplied by the free variables. Such expressions are called "linear combinations" and we will use this concept of a linear combination extensively, beginning in Section 1.7. The next example illustrates the vector form for the general solution of a nonhomogeneous system.

Example 3 Let B denote the augmented matrix for a system of linear equations

$$B = \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \end{bmatrix}.$$

Find the vector form for the general solution of the linear system.

Solution Since B is in reduced echelon form, we readily find the general solution:

$$x_1 = 3 + 2x_2 - 2x_5, x_3 = 2 + x_5, x_4 = -4 - 3x_5.$$

Expressing the general solution in vector form, we obtain

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 + 2x_2 - 2x_5 \\ x_2 \\ 2 + x_5 \\ -4 - 3x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_5 \\ 0 \\ x_5 \\ -3x_5 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

Thus, the general solution has the form $\mathbf{x} = \mathbf{b} + a\mathbf{u} + b\mathbf{v}$, where $\mathbf{b}$, $\mathbf{u}$, and $\mathbf{v}$ are fixed vectors in R^5.

Scalar Product

In vector calculus, the ***scalar product*** (or ***dot product***) of two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in R^n is defined to be the number $u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i$. For example, if

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix},$$

then the scalar product of $\mathbf{u}$ and $\mathbf{v}$ is $2(-4) + 3(2) + (-1)3 = -5$. The scalar product of two vectors will be considered further in the following section, and in Chapter 3 the properties of R^n will be more fully developed.

Matrix Multiplication

Matrix multiplication is defined in such a way as to provide a convenient mechanism for describing a linear correspondence between vectors. To illustrate, let the variables $x_1, x_2, \ldots, x_n$ and the variables $y_1, y_2, \ldots, y_m$ be related by the linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & y_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & y_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & y_m. \end{array} \qquad (1)$$