Solution The defined operations yield

$$A + B = \begin{bmatrix} 7 & 4 \\ 0 & 11 \end{bmatrix}, \quad 3C = \begin{bmatrix} 3 & 6 & -3 \\ 9 & 0 & 15 \end{bmatrix}, \quad \text{and} \quad A + 2B = \begin{bmatrix} 13 & 5 \\ 2 & 15 \end{bmatrix},$$

while $A + C$ and $B + C$ are undefined.

Vectors in R^n

Before proceeding with the definition of matrix multiplication, recall that a point in n-dimensional space is represented by an ordered n-tuple of real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Such an n-tuple will be called an ***n-dimensional vector*** and will be written in the form of a matrix,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

For example, an arbitrary three-dimensional vector has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

are distinct three-dimensional vectors. The set of all n-dimensional vectors with real components is called ***Euclidean n-space*** and will be denoted by R^n. Vectors in R^n will be denoted by boldface type. Thus R^n is the set defined by

$$R^n = \{\mathbf{x}\colon \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_1, x_2, \ldots, x_n \text{ are real numbers}\}.$$

As the notation suggests, an element of R^n can be viewed as an $(n \times 1)$ real matrix, and conversely an $(n \times 1)$ real matrix can be considered an element of R^n. Thus addition and scalar multiplication of vectors is just a special case of these operations for matrices.

Vector Form of the General Solution

Having defined addition and scalar multiplication for vectors and matrices, we can use these operations to derive a compact expression for the general solution of a consistent system of linear equations. We call this expression the ***vector form for the general solution***.

The idea of the vector form for the general solution is straightforward and is best explained by a few examples.

EXAMPLE 2 The matrix B is the augmented matrix for a homogeneous system of linear equations. Find the general solution for the linear system and express the general solution in terms of vectors

$$B = \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Solution Since B is in reduced echelon form, it is easy to write the general solution:

$$x_1 = x_3 + 3x_4, \qquad x_2 = -2x_3 - x_4.$$

In vector form, therefore, the general solution can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

This last expression is called the *vector form for the general solution.*

In general, the vector form for the general solution of a homogeneous system consists of a sum of well-determined vectors multiplied by the free variables. Such expressions are called "linear combinations" and we will use this concept of a linear combination extensively, beginning in Section 1.7. The next example illustrates the vector form for the general solution of a nonhomogeneous system.

EXAMPLE 3 Let B denote the augmented matrix for a system of linear equations

$$B = \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \end{bmatrix}.$$

Find the vector form for the general solution of the linear system.

Solution Since B is in reduced echelon form, we readily find the general solution:

$$x_1 = 3 + 2x_2 - 2x_5, x_3 = 2 + x_5, x_4 = -4 - 3x_5.$$

Expressing the general solution in vector form, we obtain

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3+2x_2-2x_5 \\ x_2 \\ 2+x_5 \\ -4-3x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_5 \\ 0 \\ x_5 \\ -3x_5 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

Thus, the general solution has the form $\mathbf{x} = \mathbf{b} + a\mathbf{u} + b\mathbf{v}$, where $\mathbf{b}$, $\mathbf{u}$, and $\mathbf{v}$ are fixed vectors in R^5. ∎

Scalar Product

In vector calculus, the ***scalar product*** (or ***dot product***) of two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in R^n is defined to be the number $u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i$. For example, if

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix},$$

then the scalar product of $\mathbf{u}$ and $\mathbf{v}$ is $2(-4) + 3(2) + (-1)3 = -5$. The scalar product of two vectors will be considered further in the following section, and in Chapter 3 the properties of R^n will be more fully developed.

Matrix Multiplication

Matrix multiplication is defined in such a way as to provide a convenient mechanism for describing a linear correspondence between vectors. To illustrate, let the variables $x_1, x_2, \ldots, x_n$ and the variables $y_1, y_2, \ldots, y_m$ be related by the linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m. \end{aligned} \qquad (1)$$

If we set

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

then (1) defines a correspondence $\mathbf{x} \to \mathbf{y}$ from vectors in R^n to vectors in R^m. The ith equation of (1) is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = y_i,$$

and this can be written in a briefer form as

$$\sum_{j=1}^{n} a_{ij}x_j = y_i. \qquad (2)$$

If A is the coefficient matrix of system (1),

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

then the left-hand side of Eq. (2) is precisely the scalar product of the ith row of A with the vector $\mathbf{x}$. Thus if we define the product of A and $\mathbf{x}$ to be the $(m \times 1)$ vector $A\mathbf{x}$ whose ith component is the scalar product of the ith row of A with $\mathbf{x}$, then $A\mathbf{x}$ is given by

$$A\mathbf{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix}.$$

Using the definition of equality (Definition 5), we see that the simple matrix equation

$$A\mathbf{x} = \mathbf{y} \qquad (3$$

is equivalent to system (1).

In a natural fashion, we can extend the idea of the product of a matrix and a vectc to the product, AB, of an $(m \times n)$ matrix A and an $(n \times s)$ matrix B by defining th ijth entry of AB to be the scalar product of the ith row of A with the jth column of E Formally, we have the following definition.

DEFINITION 8 Let $A = (a_{ij})$ be an $(m \times n)$ matrix, and let $B = (b_{ij})$ be an $(r \times s)$ matrix. If $n = r$, then the ***product*** AB is the $(m \times s)$ matrix defined by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

If $n \neq r$, then the product AB is not defined.

The definition can be visualized by referring to Fig. 1.14.

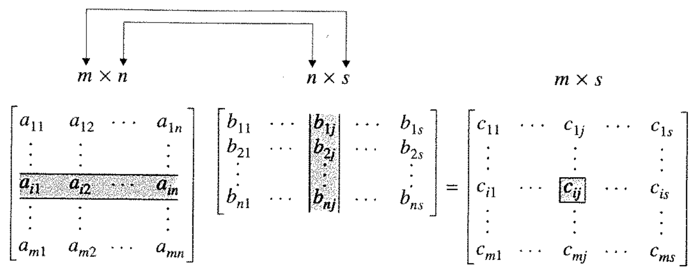

Figure 1.14 The ijth entry of AB is the scalar product of the ith row of A and the jth column of B.

Thus the product AB is defined only when the inside dimensions of A and B are equal. In this case the outside dimensions, m and s, give the size of AB. Furthermore, the ijth entry of AB is the scalar product of the ith row of A with the jth column of B. For example,

$$\begin{bmatrix} 2 & 1 & -3 \\ -2 & 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+1(0)+(-3)2 & 2(2)+1(-3)+(-3)1 \\ (-2)(-1)+2(0)+4(2) & (-2)2+2(-3)+4(1) \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 10 & -6 \end{bmatrix},$$

whereas the product

$$\begin{bmatrix} 2 & 1 & -3 \\ -2 & 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix}$$

is undefined.

EXAMPLE 4 Let the matrices A, B, C, and D be given by

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Find each of AB, BA, AC, CA, CD, and DC, or state that the indicated product is undefined.

Solution The definition of matrix multiplication yields

$$AB = \begin{bmatrix} -1 & -2 \\ -3 & -2 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 0 \\ -3 & -4 \end{bmatrix}, \quad \text{and} \quad AC = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{bmatrix}.$$

The product CA is undefined, and

$$CD = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad DC = \begin{bmatrix} 3 & 1 & -5 \\ -1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Example 4 illustrates that matrix multiplication is not commutative; that is, normally AB and BA are different matrices. Indeed, the product AB may be defined while the product BA is undefined, or both may be defined but have different dimensions. Even when AB and BA have the same size, they usually are not equal.

EXAMPLE 5 Express each of the linear systems

$$\begin{aligned} x_1 &= 2y_1 - y_2 \\ x_2 &= -3y_1 + 2y_2 \\ x_3 &= y_1 + 3y_2 \end{aligned} \quad \text{and} \quad \begin{aligned} y_1 &= -4z_1 + 2z_2 \\ y_2 &= 3z_1 + z_2 \end{aligned}$$

as a matrix equation and use matrix multiplication to express x_1, x_2, and x_3 in terms of z_1 and z_2.

Solution We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Substituting for $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in the left-hand equation gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -11 & 3 \\ 18 & -4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} x_1 &= -11z_1 + 3z_2 \\ x_2 &= 18z_1 - 4z_2 \\ x_3 &= 5z_1 + 5z_2. \end{aligned}$$

The use of the matrix equation (3) to represent the linear system (1) provides a convenient notational device for representing the $(m \times n)$ system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{4}$$

of linear equations with unknowns $x_1, \ldots, x_n$. Specifically, if $A = (a_{ij})$ is the coefficient matrix of (4), and if the unknown $(n \times 1)$ matrix $\mathbf{x}$ and the constant $(m \times 1)$ matrix $\mathbf{b}$ are defined by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then the system (4) is equivalent to the matrix equation

$$A\mathbf{x} = \mathbf{b}. \tag{5}$$

EXAMPLE 6 Solve the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & -1 \\ 2 & 8 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}.$$

Solution The matrix equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the (3×3) linear system

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ 2x_1 + 5x_2 - x_3 &= 6 \\ 2x_1 + 8x_2 - 2x_3 &= 6. \end{aligned}$$

This system can be solved in the usual way—that is, by reducing the augmented matrix—to obtain $x_1 = 2, x_2 = 1, x_3 = 3$. Therefore,

$$\mathbf{s} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

is the unique solution to $A\mathbf{x} = \mathbf{b}$.

Other Formulations of Matrix Multiplication

It is frequently convenient and useful to express an $(m \times n)$ matrix $A = (a_{ij})$ in the form

$$A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n], \tag{6}$$

where for each j, $1 \le j \le n$, $\mathbf{A}_j$ denotes the jth column of A. That is, $\mathbf{A}_j$ is the $(m \times 1)$ column vector

$$\mathbf{A}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

For example, if A is the (2×3) matrix

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 0 \end{bmatrix}, \tag{7}$$

then $A = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]$, where

$$\mathbf{A}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

The next two theorems use Eq. (6) to provide alternative ways of expressing the matrix products $A\mathbf{x}$ and AB; these methods will be extremely useful in our later development of matrix theory.

THEOREM 5 Let $A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$ be an $(m \times n)$ matrix whose jth column is $\mathbf{A}_j$, and let $\mathbf{x}$ be the $(n \times 1)$ column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then the product $A\mathbf{x}$ can be expressed as

$$A\mathbf{x} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n.$$

The proof of this theorem is not difficult and uses only Definitions 5, 6, 7, and 8; the proof is left as an exercise for the reader. To illustrate Theorem 5, let A be the matrix

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 0 \end{bmatrix},$$

and let $\mathbf{x}$ be the vector in R^3,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$