The variable x_2 has now been eliminated from the first and third equations. Next, we eliminate x_3 from the first and second equations and leave x_3, with coefficient 1, in the third equation:

System:	*Augmented Matrix:*
$(-1/3)E_3$:	$(-1/3)R_3$:
$\begin{aligned} x_1 \qquad - 5x_3 &= -3 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$	$\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
$E_1 + 5E_3$:	$R_1 + 5R_3$:
$\begin{aligned} x_1 \qquad &= 7 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$	$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
$E_2 - 2E_3$:	$R_2 - 2R_3$:
$\begin{aligned} x_1 \qquad &= 7 \\ x_2 \qquad &= -2 \\ x_3 &= 2 \end{aligned}$	$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

The last system above clearly has a unique solution given by $x_1 = 7$, $x_2 = -2$, and $x_3 = 2$. Because the final system is equivalent to the original given system, both systems have the same solution.

The reduction process used in the preceding example is known as ***Gauss-Jordan elimination*** and will be explained in Section 1.2. Note the advantage of the shorthand notation provided by matrices. Because we do not need to list the variables, the sequence of steps in the right-hand column is easier to perform and record.

Example 7 illustrates that row equivalent augmented matrices represent equivalent systems of equations. The following corollary to Theorem 1 states this in mathematical terms.

COROLLARY Suppose $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are augmented matrices, each representing a different $(m \times n)$ system of linear equations. If $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are row equivalent matrices, then the two systems are also equivalent.

1.1 EXERCISES

Which of the equations in Exercises 1–6 are linear?

1. $x_1 + 2x_3 = 3$
2. $x_1x_2 + x_2 = 1$
3. $x_1 - x_2 = \sin^2 x_1 + \cos^2 x_1$
4. $x_1 - x_2 = \sin^2 x_1 + \cos^2 x_2$
5. $|x_1| - |x_2| = 0$
6. $\pi x_1 + \sqrt{7}x_2 = \sqrt{3}$

In Exercises 7–10, coefficients are given for a system of the form (2). Display the system and verify that the given values constitute a solution.

7. $a_{11} = 1,\ a_{12} = 3,\ a_{21} = 4,\ a_{22} = -1,$ $b_1 = 7,\ b_2 = 2;\ x_1 = 1,\ x_2 = 2$

8. $a_{11} = 6,\ a_{12} = -1,\ a_{13} = 1,\ a_{21} = 1,$ $a_{22} = 2,\ a_{23} = 4,\ b_1 = 14,\ b_2 = 4;$ $x_1 = 2,\ x_2 = -1,\ x_3 = 1$
9. $a_{11} = 1,\ a_{12} = 1,\ a_{21} = 3,\ a_{22} = 4,$ $a_{31} = -1,\ a_{32} = 2,\ b_1 = 0,\ b_2 = -1,$ $b_3 = -3;\ x_1 = 1,\ x_2 = -1$
10. $a_{11} = 0,\ a_{12} = 3,\ a_{21} = 4,\ a_{22} = 0,$ $b_1 = 9,\ b_2 = 8;\ x_1 = 2,\ x_2 = 3$

In Exercises 11–14, sketch a graph for each equation to determine whether the system has a unique solution, no solution, or infinitely many solutions.

11. $2x + y = 5$; $x - y = 1$
12. $2x - y = -1$; $2x - y = 2$
13. $3x + 2y = 6$; $-6x - 4y = -12$
14. $2x + y = 5$; $x - y = 1$; $x + 3y = 9$
15. The (2×3) system of linear equations
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
is represented geometrically by two planes. How are the planes related when:
a) The system has no solution?
b) The system has infinitely many solutions?
Is it possible for the system to have a unique solution? Explain.

In Exercises 16–18, determine whether the given (2×3) system of linear equations represents coincident planes (that is, the same plane), two parallel planes, or two planes whose intersection is a line. In the latter case, give the parametric equations for the line; that is, give equations of the form $x = at + b$, $y = ct + d$, $z = et + f$.

16. $2x_1 + x_2 + x_3 = 3$; $-2x_1 + x_2 - x_3 = 1$
17. $x_1 + 2x_2 - x_3 = 2$; $x_1 + x_2 + x_3 = 3$
18. $x_1 + 3x_2 - 2x_3 = -1$; $2x_1 + 6x_2 - 4x_3 = -2$
19. Display the (2×3) matrix $A = (a_{ij})$, where $a_{11} = 2$, $a_{12} = 1$, $a_{13} = 6$, $a_{21} = 4$, $a_{22} = 3$, and $a_{23} = 8$.
20. Display the (2×4) matrix $C = (c_{ij})$, where $c_{23} = 4$, $c_{12} = 2$, $c_{21} = 2$, $c_{14} = 1$, $c_{22} = 2$, $c_{24} = 3$, $c_{11} = 1$, and $c_{13} = 7$.
21. Display the (3×3) matrix $Q = (q_{ij})$, where $q_{23} = 1$, $q_{32} = 2$, $q_{11} = 1$, $q_{13} = -3$, $q_{22} = 1$, $q_{33} = 1$, $q_{21} = 2$, $q_{12} = 4$, and $q_{31} = 3$.
22. Suppose the matrix C in Exercise 20 is the augmented matrix for a system of linear equations. Display the system.
23. Repeat Exercise 22 for the matrices in Exercises 19 and 21.

In Exercises 24–29, display the coefficient matrix A and the augmented matrix B for the given system.

24. $x_1 - x_2 = -1$; $x_1 + x_2 = 3$
25. $x_1 + x_2 - x_3 = 2$; $2x_1 \quad - x_3 = 1$
26. $x_1 + 3x_2 - x_3 = 1$; $2x_1 + 5x_2 + x_3 = 5$; $x_1 + x_2 + x_3 = 3$
27. $x_1 + x_2 + 2x_3 = 6$; $3x_1 + 4x_2 - x_3 = 5$; $-x_1 + x_2 + x_3 = 2$
28. $x_1 + x_2 - 3x_3 = -1$; $x_1 + 2x_2 - 5x_3 = -2$; $-x_1 - 3x_2 + 7x_3 = 3$
29. $x_1 + x_2 + x_3 = 1$; $2x_1 + 3x_2 + x_3 = 2$; $x_1 - x_2 + 3x_3 = 2$

In Exercises 30–36, display the augmented matrix for the given system. Use elementary operations on equations to obtain an equivalent system of equations in which x_1 appears in the first equation with coefficient one and has been eliminated from the remaining equations. Simultaneously, perform the corresponding elementary row operations on the augmented matrix.

30. $2x_1 + 3x_2 = 6$; $4x_1 - x_2 = 7$
31. $x_1 + 2x_2 - x_3 = 1$; $x_1 + x_2 + 2x_3 = 2$; $-2x_1 + x_2 \quad = 4$
32. $x_2 + x_3 = 4$; $x_1 - x_2 + 2x_3 = 1$; $2x_1 + x_2 - x_3 = 6$
33. $x_1 + x_2 = 9$; $x_1 - x_2 = 7$; $3x_1 + x_2 = 6$
34. $x_1 + x_2 + x_3 - x_4 = 1$; $-x_1 + x_2 - x_3 + x_4 = 3$; $-2x_1 + x_2 + x_3 - x_4 = 2$
35. $x_2 + x_3 - x_4 = 3$; $x_1 + 2x_2 - x_3 + x_4 = 1$; $-x_1 + x_2 + 7x_3 - x_4 = 0$
36. $x_1 + x_2 = 0$; $x_1 - x_2 = 0$; $3x_1 + x_2 = 0$
37. Consider the equation $2x_1 - 3x_2 + x_3 - x_4 = 3$.
a) In the six different possible combinations, set any two of the variables equal to 1 and graph the equation in terms of the other two.
b) What type of graph do you always get when you set two of the variables equal to two fixed constants?
c) What is one possible reason the equation in formula (1) is called *linear*?

38. Consider the (2×2) system

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Show that if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then this system is equivalent to a system of the form

$$c_{11}x_1 + c_{12}x_2 = d_1$$
$$c_{22}x_2 = d_2,$$

where $c_{11} \neq 0$ and $c_{22} \neq 0$. Note that the second system always has a solution. [*Hint:* First suppose that $a_{11} \neq 0$, and then consider the special case in which $a_{11} = 0$.]

39. In the following (2×2) linear systems (A) and (B), c is a nonzero scalar. Prove that any solution, $x_1 = s_1$, $x_2 = s_2$, for (A) is also a solution for (B). Conversely, show that any solution, $x_1 = t_1$, $x_2 = t_2$, for (B) is also a solution for (A). Where is the assumption that c is nonzero required?

$$\text{(A)} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

$$\text{(B)} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ ca_{21}x_1 + ca_{22}x_2 &= cb_2 \end{aligned}$$

40. In the (2×2) linear systems that follow, the system (B) is obtained from (A) by performing the elementary operation $E_2 + cE_1$. Prove that any solution, $x_1 = s_1$, $x_2 = s_2$, for (A) is a solution for (B). Similarly, prove that any solution, $x_1 = t_1$, $x_2 = t_2$, for (B) is a solution for (A).

$$\text{(A)} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

$$\text{(B)} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ (a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 &= b_2 + cb_1 \end{aligned}$$

41. Prove that any of the elementary operations in Theorem 1 applied to system (2) produces an equivalent system. [*Hint:* To simplify this proof, represent the ith equation in system (2) as $f_i(x_1, x_2, \ldots, x_n) = b_i$; so

$$f_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

for $i = 1, 2, \ldots, m$. With this notation, system (2) has the form of (A), which follows. Next, for example, if a multiple of c times the jth equation is added to the kth equation, a new system of the form (B) is produced:

$$\begin{array}{cc} \textbf{(A)} & \textbf{(B)} \\ f_1(x_1, x_2, \ldots, x_n) = b_1 & f_1(x_1, x_2, \ldots, x_n) = b_1 \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ f_j(x_1, x_2, \ldots, x_n) = b_j & f_j(x_1, x_2, \ldots, x_n) = b_j \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ f_k(x_1, x_2, \ldots, x_n) = b_k & g(x_1, x_2, \ldots, x_n) = r \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ f_m(x_1, x_2, \ldots, x_n) = b_m & f_m(x_1, x_2, \ldots, x_n) = b_m \end{array}$$

where $g(x_1, x_2, \ldots, x_n) = f_k(x_1, x_2, \ldots, x_n) + cf_j(x_1, x_2, \ldots, x_n)$, and $r = b_k + cb_j$. To show that the operation gives an equivalent system, show that any solution for (A) is a solution for (B), and vice versa.]

42. Solve the system of two nonlinear equations in two unknowns

$$\begin{aligned} x_1^2 - 2x_1 + x_2^2 &= 3 \\ x_1^2 \qquad - x_2^2 &= 1. \end{aligned}$$

1.2 ECHELON FORM AND GAUSS-JORDAN ELIMINATION

As we noted in the previous section, our method for solving a system of linear equations will be to pass to the augmented matrix, use elementary row operations to reduce the augmented matrix, and then solve the simpler but equivalent system represented by the reduced matrix. This procedure is illustrated in Fig. 1.3.

The objective of the Gauss-Jordan reduction process (represented by the middle block in Fig. 1.3) is to obtain a system of equations simplified to the point where we

Given system of equations → Augmented matrix → Reduced matrix → Reduced system of equations → Solution

Figure 1.3 Procedure for solving a system of linear equations

can immediately describe the solution. See, for example, Examples 6 and 7 in Section 1.1. We turn now to the question of how to describe this objective in mathematical terms—that is, how do we know when the system has been simplified as much as it can be? The answer is: The system has been simplified as much as possible when it is in reduced echelon form.

Echelon Form

When an augmented matrix is reduced to the form known as *echelon form*, it is easy to solve the linear system represented by the reduced matrix. The formal description of echelon form is given in Definition 3. Then, in Definition 4, we describe an even simpler form known as *reduced echelon form*.

DEFINITION 3 An $(m \times n)$ matrix B is in ***echelon form*** if:

1. All rows that consist entirely of zeros are grouped together at the bottom of the matrix.
2. In every nonzero row, the first nonzero entry (counting from left to right) is a 1.
3. If the $(i + 1)$-st row contains nonzero entries, then the first nonzero entry is in a column to the right of the first nonzero entry in the ith row.

Put informally, a matrix A is in echelon form if the nonzero entries in A form a staircase-like pattern, such as the four examples shown in Fig. 1.4. (***Note:*** Exercise 46 shows that there are exactly seven different types of echelon form for a (3×3) matrix. Figure 1.4 illustrates four of the possible patterns. In Fig. 1.4, the entries marked $*$ can be zero or nonzero.)

$$A = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure 1.4 Patterns for four of the seven possible types of (3×3) matrices in echelon form. Entries marked $*$ can be either zero or nonzero.