1.5 EXERCISES

The (2×2) matrices listed in Eq. (9) are used in several of the exercises that follow.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix},$$
$$C = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (9)$$

Exercises 1–6 refer to the matrices in Eq. (9).

1. Find (a) $A+B$; (b) $A+C$; (c) $6B$; and (d) $B+3C$.

2. Find (a) $B+C$; (b) $3A$; (c) $A+2C$; and (d) $C+8Z$.

3. Find a matrix D such that $A+D=B$.

4. Find a matrix D such that $A+2D=C$.

5. Find a matrix D such that $A+2B+2D=3B$.

6. Find a matrix D such that $2A+5B+D=2B+3A$.

The vectors listed in Eq. (10) are used in several of the exercises that follow.

$$\mathbf{r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$
$$\mathbf{t} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \qquad (10)$$

In Exercises 7–12, perform the indicated computation, using the vectors in Eq. (10) and the matrices in Eq. (9).

7. a) $\mathbf{r}+\mathbf{s}$
b) $2\mathbf{r}+\mathbf{t}$
c) $2\mathbf{s}+\mathbf{u}$

8. a) $\mathbf{t}+\mathbf{s}$
b) $\mathbf{r}+3\mathbf{u}$
c) $2\mathbf{u}+3\mathbf{t}$

9. a) $A\mathbf{r}$
b) $B\mathbf{r}$
c) $C(\mathbf{s}+3\mathbf{t})$

10. a) $B\mathbf{t}$
b) $C(\mathbf{r}+\mathbf{s})$
c) $B(\mathbf{r}+\mathbf{s})$

11. a) $(A+2B)\mathbf{r}$
b) $(B+C)\mathbf{u}$

12. a) $(A+C)\mathbf{r}$
b) $(2B+3C)\mathbf{s}$

Exercises 13–20 refer to the vectors in Eq. (10). In each exercise, find scalars a_1 and a_2 that satisfy the given equation, or state that the equation has no solution.

13. $a_1\mathbf{r}+a_2\mathbf{s}=\mathbf{t}$

14. $a_1\mathbf{r}+a_2\mathbf{s}=\mathbf{u}$

15. $a_1\mathbf{s}+a_2\mathbf{t}=\mathbf{u}$

16. $a_1\mathbf{s}+a_2\mathbf{t}=\mathbf{r}+\mathbf{t}$

17. $a_1\mathbf{s}+a_2\mathbf{u}=2\mathbf{r}+\mathbf{t}$

18. $a_1\mathbf{s}+a_2\mathbf{u}=\mathbf{t}$

19. $a_1\mathbf{t}+a_2\mathbf{u}=3\mathbf{s}+4\mathbf{t}$

20. $a_1\mathbf{t}+a_2\mathbf{u}=3\mathbf{r}+2\mathbf{s}$

Exercises 21–24 refer to the matrices in Eq. (9) and the vectors in Eq. (10).

21. Find $\mathbf{w}_2$, where $\mathbf{w}_1 = B\mathbf{r}$ and $\mathbf{w}_2 = A\mathbf{w}_1$. Calculate $Q = AB$. Calculate $Q\mathbf{r}$ and verify that $\mathbf{w}_2$ is equal to $Q\mathbf{r}$.

22. Find $\mathbf{w}_2$, where $\mathbf{w}_1 = C\mathbf{s}$ and $\mathbf{w}_2 = A\mathbf{w}_1$. Calculate $Q = AC$. Calculate $Q\mathbf{s}$ and verify that $\mathbf{w}_2$ is equal to $Q\mathbf{s}$.

23. Find $\mathbf{w}_3$, where $\mathbf{w}_1 = C\mathbf{r}$, $\mathbf{w}_2 = B\mathbf{w}_1$, and $\mathbf{w}_3 = A\mathbf{w}_2$. Calculate $Q = A(BC)$ and verify that $\mathbf{w}_3$ is equal to $Q\mathbf{r}$.

24. Find $\mathbf{w}_3$, where $\mathbf{w}_1 = A\mathbf{r}$, $\mathbf{w}_2 = C\mathbf{w}_1$, and $\mathbf{w}_3 = B\mathbf{w}_2$. Calculate $Q = B(CA)$ and verify that $\mathbf{w}_3$ is equal to $Q\mathbf{r}$.

Exercises 25–30 refer to the matrices in Eq. (9). Find each of the following.

25. $(A+B)C$

26. $(A+2B)A$

27. $(A+C)B$

28. $(B+C)Z$

29. $A(BZ)$

30. $Z(AB)$

The matrices and vectors listed in Eq. (11) are used in several of the exercises that follow.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$
$$\mathbf{v} = [\,2, 4\,], \quad C = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 8 & -1 \\ 3 & 2 \end{bmatrix}, \qquad (11)$$
$$D = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 2 & 0 & 0 & 4 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 1 & 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

Exercises 31–41 refer to the matrices and vectors in Eq. (11). Find each of the following.

31. AB and BA

32. DC

33. $A\mathbf{u}$ and $\mathbf{v}A$

34. $\mathbf{uv}$ and $\mathbf{vu}$

35. $\mathbf{v}(B\mathbf{u})$

36. $B\mathbf{u}$

37. CA

38. CB

39. $C(B\mathbf{u})$

40. $(AB)\mathbf{u}$ and $A(B\mathbf{u})$

41. $(BA)\mathbf{u}$ and $B(A\mathbf{u})$

In Exercises 42–49, the given matrix is the augmented matrix for a system of linear equations. Give the vector form for the general solution.

42. $\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

44. $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

45. $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{bmatrix}$

47. $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{bmatrix}$

48. $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

49. $\begin{bmatrix} 1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

50. In Exercise 40, the calculations $(AB)\mathbf{u}$ and $A(B\mathbf{u})$ produce the same result. Which calculation requires fewer multiplications of individual matrix entries? (For example, it takes two multiplications to get the (1, 1) entry of AB.)

51. The next section will show that all the following calculations produce the same result:

$$C[A(B\mathbf{u})] = (CA)(B\mathbf{u}) = [C(AB)]\mathbf{u} = C[(AB)\mathbf{u}].$$

Convince yourself that the first expression requires the fewest individual multiplications. [*Hint:* Forming $B\mathbf{u}$ takes four multiplications, and thus $A(B\mathbf{u})$ takes eight multiplications, and so on.] Count the number of multiplications required for each of the four preceding calculations.

52. Refer to the matrices and vectors in Eq. (11).

a) Identify the column vectors in $A = [\mathbf{A}_1, \mathbf{A}_2]$ and $D = [\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4]$.

b) In part (a), is $\mathbf{A}_1$ in R^2, R^3, or R^4? Is $\mathbf{D}_1$ in R^2, R^3, or R^4?

c) Form the (2×2) matrix with columns $[A\mathbf{B}_1, A\mathbf{B}_2]$, and verify that this matrix is the product AB.

d) Verify that the vector $D\mathbf{w}$ is the same as $2\mathbf{D}_1 + 3\mathbf{D}_2 + \mathbf{D}_3 + \mathbf{D}_4$.

53. Determine whether the following matrix products are defined. When the product is defined, give the size of the product.

a) AB and BA, where A is (2×3) and B is (3×4)

b) AB and BA, where A is (2×3) and B is (2×4)

c) AB and BA, where A is (3×7) and B is (6×3)

d) AB and BA, where A is (2×3) and B is (3×2)

e) AB and BA, where A is (3×3) and B is (3×1)

f) $A(BC)$ and $(AB)C$, where A is (2×3), B is (3×5), and C is (5×4)

g) AB and BA, where A is (4×1) and B is (1×4)

54. What is the size of the product $(AB)(CD)$, where A is (2×3), B is (3×4), C is (4×4), and D is (4×2)? Also calculate the size of $A[B(CD)]$ and $[(AB)C]D$.

55. If A is a matrix, what should the symbol A^2 mean? What restrictions on A are required in order that A^2 be defined?

56. Set

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and}$$
$$B = \begin{bmatrix} 1 & b \\ b^{-1} & 1 \end{bmatrix},$$

where $b \neq 0$. Show that O, A, and B are solutions to the matrix equation $X^2 - 2X = O$. Conclude that this quadratic equation has infinitely many solutions.

57. Two newspapers compete for subscriptions in a region with 300,000 households. Assume that no household subscribes to both newspapers and that the following table gives the probabilities that a household will change its subscription status during the year.

	From A	*From B*	*From None*
To A	.70	.15	.30
To B	.20	.80	.20
To None	.10	.05	.50

For example, an interpretation of the first column of the table is that during a given year, newspaper A can expect to keep 70% of its current subscribers while losing 20% to newspaper B and 10% to no subscription.

At the beginning of a particular year, suppose that 150,000 households subscribe to newspaper A, 100,000 subscribe to newspaper B, and 50,000 have no subscription. Let P and $\mathbf{x}$ be defined by

$$P = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 150{,}000 \\ 100{,}000 \\ 50{,}000 \end{bmatrix}.$$

The vector $\mathbf{x}$ is called the ***state vector*** for the beginning of the year. Calculate $P\mathbf{x}$ and $P^2\mathbf{x}$ and interpret the resulting vectors.

58. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

a) Find all matrices $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = BA$.

b) Use the results of part (a) to exhibit (2×2) matrices B and C such that $AB = BA$ and $AC \neq CA$.

59. Let A and B be matrices such that the product AB is defined and is a square matrix. Argue that the product BA is also defined and is a square matrix.

60. Let A and B be matrices such that the product AB is defined. Use Theorem 6 to prove each of the following.

a) If B has a column of zeros, then so does AB.

b) If B has two identical columns, then so does AB.

61. a) Express each of the linear systems i) and ii) in the form $A\mathbf{x} = \mathbf{b}$.

i)
$$\begin{aligned} 2x_1 - x_2 &= 3 \\ x_1 + x_2 &= 3 \end{aligned}$$

ii)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \end{aligned}$$

b) Express systems i) and ii) in the form of Eq. (8).

c) Solve systems i) and ii) by Gaussian elimination. For each system $A\mathbf{x} = \mathbf{b}$, represent $\mathbf{b}$ as a linear combination of the columns of the coefficient matrix.

62. Solve $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

63. Let A and I be the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a) Find a (2×2) matrix B such that $AB = I$. [*Hint:* Use Theorem 6 to determine the column vectors of B.]

b) Show that $AB = BA$ for the matrix B found in part (a).

64. Prove Theorem 5 by showing that the ith component of $A\mathbf{x}$ is equal to the ith component of $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n$, where $1 \leq i \leq m$.

65. For A and C, which follow, find a matrix B (if possible) such that $AB = C$.

a) $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 6 \\ 3 & 6 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$

where $B \neq C$.

66. A (3×3) matrix $T = (t_{ij})$ is called an ***upper-triangular*** matrix if T has the form

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{bmatrix}.$$

Formally, T is upper triangular if $t_{ij} = 0$ whenever $i > j$. If A and B are upper-triangular (3×3) matrices, verify that the product AB is also upper triangular.

67. An $(n \times n)$ matrix $T = (t_{ij})$ is called upper triangular if $t_{ij} = 0$ whenever $i > j$. Suppose that A and B are $(n \times n)$ upper-triangular matrices. Use Definition 8 to prove that the product AB is upper triangular. That is, show that the ijth entry of AB is zero when $i > j$.

In Exercises 68–70, find the vector form for the general solution.

68.
$$\begin{aligned} x_1 + 3x_2 - 3x_3 + 2x_4 - 3x_5 &= -4 \\ 3x_1 + 9x_2 - 10x_3 + 10x_4 - 14x_5 &= 2 \\ 2x_1 + 6x_2 - 10x_3 + 21x_4 - 25x_5 &= 53 \end{aligned}$$

69.
$$\begin{aligned} 14x_1 - 8x_2 + 3x_3 - 49x_4 + 29x_5 &= 44 \\ -8x_1 + 5x_2 - 2x_3 + 29x_4 - 16x_5 &= -24 \\ 3x_1 - 2x_2 + x_3 - 11x_4 + 6x_5 &= 9 \end{aligned}$$

70.
$$\begin{aligned} 18x_1 + 18x_2 - 10x_3 + 7x_4 + 2x_5 + 50x_6 &= 26 \\ -10x_1 - 10x_2 + 6x_3 - 4x_4 - x_5 - 27x_6 &= -13 \\ 7x_1 + 7x_2 - 4x_3 + 5x_4 + 2x_5 + 30x_6 &= 18 \\ 2x_1 + 2x_2 - x_3 + 2x_4 + x_5 + 12x_6 &= 8 \end{aligned}$$

71. In Exercise 57 we saw that the state vector giving the number of newspaper subscribers in year n could be found by forming $P^n\mathbf{x}$ where $\mathbf{x}$ is the initial state. Later, in Section 3.8, we will see that as n grows larger and larger, the vector $P^n\mathbf{x}$ tends toward a limit. Use MATLAB to calculate $P^n\mathbf{x}$ for $n = 1, 2, \ldots, 30$. For ease of reading, display the results using bank format in the MATLAB numeric options menu. What do you think the steady state distribution of newspapers will be?

1.6 ALGEBRAIC PROPERTIES OF MATRIX OPERATIONS

In the previous section we defined the matrix operations of addition, multiplication, and the multiplication of a matrix by a scalar. For these operations to be useful, the basic rules they obey must be determined. As we will presently see, many of the familiar algebraic properties of real numbers also hold for matrices. There are, however, important exceptions. We have already noted, for example, that matrix multiplication is not commutative. Another property of real numbers that does not carry over to matrices is the cancellation law for multiplication. That is, if a, b, and c are real numbers such that $ab = ac$ and $a \neq 0$, then $b = c$. By contrast, consider the three matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

Note that $AB = AC$ but $B \neq C$. This example shows that the familiar cancellation law for real numbers does not apply to matrix multiplication.

Properties of Matrix Operations

The next three theorems list algebraic properties that do hold for matrix operations. In some cases, although the rule seems obvious and the proof simple, certain subtleties should be noted. For example, Theorem 9 asserts that $(r + s)A = rA + sA$, where r and s are scalars and A is an $(m \times n)$ matrix. Although the same addition symbol, $+$, appears on both sides of the equation, two different addition operations are involved; $r + s$ is the sum of two scalars, and $rA + sA$ is the sum of two matrices.

Our first theorem lists some of the properties satisfied by matrix addition.

THEOREM 7 If A, B, and C are $(m \times n)$ matrices, then the following are true:

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$.
3. There exists a unique $(m \times n)$ matrix $\mathcal{O}$ (called the ***zero matrix***) such that $A + \mathcal{O} = A$ for every $(m \times n)$ matrix A.
4. Given an $(m \times n)$ matrix A, there exists a unique $(m \times n)$ matrix P such that $A + P = \mathcal{O}$.