Sec. 1.2 Echelon form, Reduced echelon form, and Gauss-Jordan elimination.

(1) Generalization of Ex. 7 of Sec. 1.1 into a general algorithm.

The steps of Ex. 7 can be combined into two groups.

1) Using EOs (or EROs for the augmented matrix), reduce the original l.s. (or the augmented matrix) to the form:

$$
\begin{aligned}
1\cdot x_1 + (*)\cdot x_2 + (*)\cdot x_3 &= (*)\\
1\cdot x_2 + (*)\cdot x_3 &= (*)\\
1\cdot x_3 &= (*)
\end{aligned}
$$

This form is obtained doing EOs from top to bottom

2) Then use back-substitution:
- first, substitute E_3 into E_2 & E_1;
- then, substitute E_2 into E_1.

The result is:

$$
\begin{aligned}
1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= (*)\\
1\cdot x_2 + 0\cdot x_3 &= (*)\\
1\cdot x_3 &= (*),
\end{aligned}
$$

This form is obtained doing EOs from bottom to top.

from which you immediately deduce the solution to the l.s.!

These groups of steps are reviewed in the notes on Ex. 7 of Sec. 1.1 posted next to Sec. 1.2.

These groups of steps can be shown <u>schematically</u> as:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

original augmented matrix	an example of Echelon Form (EF)	an example of Reduced Echelon Form (REF)

More generally, EF & REF are defined as follows.

EF
$$\begin{pmatrix} \boxed{1} & * & * & * & * \\ 0 & \boxed{1} & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 1st <u>nonzero</u> entry in each row is $\boxed{1}$
- Nonzero rows are arranged in a staircase-like form.
- Zero rows are grouped at the bottom.

REF
$$\begin{pmatrix} \boxed{1} & 0 & * & 0 & * \\ 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Same as EF, plus: the 1st nonzero entry in each row is the <u>only</u> nonzero entry in its column.

(These are (1st) first entries in their rows, so they <u>can</u> be nonzero.)

All possible EF's of a 3×3 matrix are listed on p. 15;

All possible REF's of a 3×3 matrix are listed on p. 16.

MUST STUDY THESE EXAMPLES on your own.
See also Ex. 1 in book.

Ex. 1 We will now solve a l.s. similar to that in Ex. 7 of Sec. 1.1, using only the EROs (no EOs) and the concepts of EF & REF.

l.s.:

$$\begin{aligned} 2x_1 \qquad\quad - 3x_3 - x_4 &= -3 \\ -x_1 + 2x_2 \qquad\quad + 4x_4 &= 0 \\ x_2 + x_3 \qquad\quad &= 1 \\ x_1 + x_2 - x_3 \qquad\quad &= -1 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 0 & -3 & -1 & -3 \\ -1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 \end{array}\right)$$

We'll repeat the steps of Ex. 7 of Sec. 1.1 (in the "rewritten form", see pp. 1-12a,b).

- See also Ex. 3 & 4 of Sec. 1.2 in book for the same process,
- as well as the website posted under the link for notes for Sec. 1.2 on the course webpage.

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Augmented Matrix	ERO leading to next stage	Comment about ERO
$\left(\begin{array}{cccc\|c} 2 & 0 & -3 & -1 & -3 \\ -1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 \end{array}\right)$	$R_4 \leftrightarrow R_1$	Obtain a leading $\boxed{1}$ in R_1.
$\Downarrow$		
$\left(\begin{array}{cccc\|c} 1 & 1 & -1 & 0 & -1 \\ -1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & -3 & -1 & -3 \end{array}\right)$	$R_2 + R_1 \to R_2$ $R_4 - 2R_1 \to R_4$	Eliminate x_1 from R_2 & R_4 (<u>all</u> rows <u>below</u> R_1).
$\Downarrow$		
$\left(\begin{array}{cccc\|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 4 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -2 & -1 & -1 & -1 \end{array}\right)$	$R_2 \leftrightarrow R_3$	1) Do not use R_1 until you get to EF. 2) Obtain a leading $\boxed{1}$ in R_2.
$\Downarrow$		
$\left(\begin{array}{cccc\|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 4 & -1 \\ 0 & -2 & -1 & -1 & -1 \end{array}\right)$	$R_3 - 3R_2 \to R_3$ $R_4 + 2R_2 \to R_4$	Eliminate x_2 from <u>all</u> rows <u>below</u> R_2.

$$\Downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -4 & 4 & -4 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right) \quad -\tfrac{1}{4}R_3 \to R_3$$

1) Obtain a leading $\boxed{1}$ in R_3.
2) Do not use R_2 until you get to EF.

$$\Downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right) \quad R_4 - R_3 \to R_4$$

Eliminate X_3 from <u>all</u> rows <u>below</u> R_3.

$$\Downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_2 - R_3 \to R_2 \\ R_1 + R_3 \to R_1 \end{array}$$

1) This is EF!
2) You can now begin backsubstitution: use R_3 to eliminate X_3 from <u>all</u> rows <u>above</u> R_3.
3) The fact that $R_4 = 0$ means in the original l.s., one equation was a lin. combination of the other ones.
4) Do not use R_2 until you're done using R_3!

$$\Downarrow$$

$$\Downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 - R_2 \to R_1$$

Use R_2 to eliminate X_2 from <u>all</u> rows <u>above</u> R_2 (i.e., from R_1).

$$\Downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is REF.

Deduce the sol'n to the l.s.:

$$\left.\begin{array}{l} x_1 - 2x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - x_4 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 2x_4 \\ x_2 = -x_4 \\ x_3 = 1 + x_4, \\ x_4 = \text{free variable} \end{array}$$

Done!

② <u>Consistent & inconsistent l.s.</u>

<u>Ex. 2</u> The given matrix B is the augmented matrix of a l.s. put in REF. Write down the solution to the l.s.

(a) $$B = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Downarrow$$

$$\left.\begin{array}{l} x_1 + 2x_2 + 4x_4 = 5 \\ x_3 + \quad 6x_4 = 7 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 5 - 2x_2 - 4x_4 \\ x_3 = 7 - 6x_4 \,; \end{array}$$

x_2, x_4 = free variables

(b) $$B = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 5 - 4x_4 \\ x_3 = 7 - 6x_4 \end{array}$$

x_4 = free.

However, there is no restriction on x_2, $\Rightarrow$ x_2 is <u>also</u> free.

Avoid the <u>common mistake</u>: $\boxed{\text{no equation for } x_2 \Rightarrow x_2 = 0}$

WRONG!!!

(c) $$B = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The last eqn. says: $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = \underline{\underline{1}}$.

This cannot occur, $\Rightarrow$ this l.s., as well as the original l.s., is inconsistent.

(has <u>NO</u> solutions).

General fact:

let $[A|\underline{b}]$ be the augmented matrix of a l.s. If we put it in EF, and the last row becomes

$$[0, 0, \dots 0 \mid 1],$$

then this l.s. has no solution (is inconsistent).

Note 1: In such a case, it suffices to stop at the EF and not go to REF, because EF & REF have the same bottom row.

Note 2: In all other cases (i.e. when the l.s. is consistent (= has solutions), you must not stop at the EF but go all the way to REF.

HW: 3, 5, 7, 9, 10, 13, 15, 17, 21, 23, 27, 29, 31, 37, 41

EC #1 : 49, 51 ← must provide complete and detailed explanation of setup + details of solution by REF (otherwise, no credit, since brief sol'ns are in the Manual)

56 ← this is similar to Ex. 5 in book.