Sec. 1.3 Consistent linear systems

① Number of solutions of a l.s. Free variables.

Q: Suppose that we've put the augm. matrix of a l.s. into the REF. Can we tell, from the REF, how many soln's (i.e., 1, 0, or ∞) the l.s. has? Let's look at earlier examples...

Ex. 1(a) (= Ex. 7 of Sec. 1.1)

The REF was: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$; sol'n: $x_1 = 2$, $x_2 = -1$, $x_3 = -1$.

Each variable was defined uniquely, $\Rightarrow$ this l.s. has 1 sol'n.

Ex. 1(b) (= Ex. 2(a) of Sec. 1.2)

REF: $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 + 4x_4 = 5 \\ x_3 + 6x_4 = 7 \end{array}$

$$\Rightarrow \begin{cases} x_1 = 5 - 2x_2 - 4x_4 \\ x_3 = 7 - 6x_4 \end{cases}, \quad x_2, x_4 = \text{free variables.}$$

Since x_2, x_4 can take on ∞ many values, this l.s. has ∞ many soln's.

Observation: The "free variables" appeared because there were more variables than eqs.

Q: Is it always true that when a l.s. has more variables than eqs., it has ∞ many soln's?

Ex. 1(c) ($\approx$ Ex. 2(c) of Sec. 1.2)

REF: $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$. As we discussed in Sec. 1.2, this l.s. has no solutions!

So, here is the General Fact (= Corollary to Thm. 3 in book):
Consider a $m \times n$ l.s. with $m < n$. Then this l.s. is either inconsistent (Ex. 1(c)) or has ∞ many soln's (Ex. 1(b)).

Ex. 2 How many soln's does this l.s. have:

$$2x_1 - x_2 + x_3 = 0$$
$$x_1 + 3x_2 - x_3 = 0 \quad ?$$

Sol'n: 1) $m=2$, $n=3$, $\Rightarrow$ by the General Fact, it has either 0 or ∞ many soln's.

2) By inspection, $x_1 = x_2 = x_3 = 0$ is a sol'n! Thus, this l.s. has ∞ many soln's.

(2) Homogeneous l.s.

The l.s. in Ex. 2 was an example of a homogeneous l.s. (the r.h.s. are all 0).

The general form of a homogeneous l.s. is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n &= 0 \end{aligned}$$

all are 0's !

A homog. l.s. always has the trivial sol'n $x_1 = x_2 = \dots = x_n = 0$. This sol'n can be either the only sol'n, or one of ∞ many. Combining this with the General Fact above, we get:

Thm. 4: If a l.s. is:
- homogeneous;
- $m \times n$ with $m < n$, then

it has ∞ many sol'ns.

Note about the name:

1) "A l.s. has the zero solution" means that it has the trivial sol'n $x_1 = x_2 = \dots = x_n = 0$.

2) "A l.s. has zero solutions" means that it has no solutions (i.e., is inconsistent).

3 Application of homogeneous l.s. to curve fitting

Ex. 3 Use linear algebra to obtain the eq. of a straight line that passes through two given points $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (2, 3)$.

Sol'n: The most general form of the eq. of a straight line is

$$a \cdot x + b \cdot y + c = 0, \quad (*)$$

where (x, y) are coordinates of points on the line and (a, b, c) are the coefficients to be found.

1) @ $(x_1, y_1) = (1, 2)$: $\quad a \cdot 1 + b \cdot 2 + c \cdot 1 = 0$

@ $(x_2, y_2) = (2, 3)$: $\quad a \cdot 2 + b \cdot 3 + c \cdot 1 = 0$

This is a (2×3) homogeneous l.s. for (a, b, c), so by thm. 4 above it has ∞ many sol'n's. But how can this result in only one line that passes through 2 points ??

2) Solve the above l.s. by REF:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right) \xrightarrow{REF} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \Rightarrow$$

$$\begin{array}{l} a - c = 0 \\ b + c = 0 \end{array}, \; c = \text{free}, \Rightarrow \; a = c, \; b = -c, \; c = \text{free}.$$

3) Substitute this back into Eq. $(*)$:

$$\underset{c}{a} \cdot x + \underset{-c}{b} \cdot y + c = 0 \quad \Rightarrow$$

$$c \cdot x - c \cdot y + c = 0.$$

Since c = free, it is not necessarily 0, $\Rightarrow$ we can divide the above eq. by c:

$$x - y + 1 = 0, \quad \text{or} \quad \boxed{y = x + 1}.$$

Now suppose that we want to fit a "conic section" (= parabola, hyperbola, ellipse (circle)) through some points. 2 questions:

1) How many points uniquely define a conic sec.?

2) How to find the coefficients of the conic section?

The general eq. of a conic section is:

$$a \cdot x^2 + bxy + cy^2 + d \cdot x + e \cdot y + f = 0,$$

where a through f are 6 constants to be found.

The number of points that uniquely determine a conic section is then $6 - 1 = 5$.

For the rest of the sol'n, you

$\boxed{\text{MUST SEE EX. 9 in TEXTBOOK}}$.

<u>HW</u>: 1, 3, 5, 6;

19, 21, 25, 27, 29, 33.