Sec. 1.5   Matrix operations

① Elementary operations

Def: Let A be $m \times n$ matrix; and let B be $r \times s$ matrix. They are said to be equal of:

• their dimensions are the same $(m = r, n = s)$;

• their corresponding entries are equal.

Ex. 1 (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ (second condition is violated: order of entries matters!)

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix}$ (first condition is violated: the dimensions do not match)

One adds matrices similarly how one adds scalars.

Def: If A and B are matrices of the same dimensions, then one finds $(A+B)$ simply by adding their corresponding entries.

• Note: One cannot add matrices ef their dimensions are not equal.

Ex. 2 (a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \end{pmatrix}$,

$C = \begin{pmatrix} 21 & 22 \\ 23 & 24 \\ 25 & 26 \end{pmatrix}$.

$$A + B = \begin{pmatrix} 1+11 & 2+12 & 3+13 \\ 4+14 & 5+15 & 6+16 \end{pmatrix},$$

but cannot compute $A+C$ and $B+C$.

(b) Find D s.t. $A+D=B$.

Solution: $D = B - A = \begin{pmatrix} 11-1 & 12-2 & 13-3 \\ 14-4 & 15-5 & 16-6 \end{pmatrix}$.

Def: To multiply matrix A by scalar r, simply multiply _each_ entry of A by r.
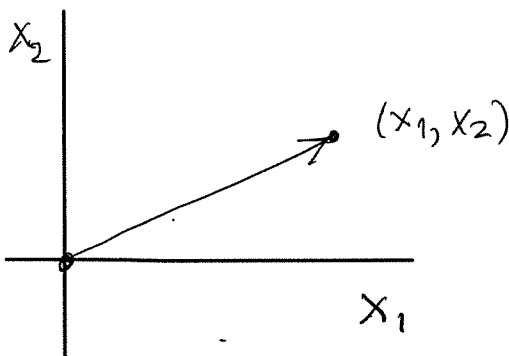
Ex. 3

$$11 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 22 \\ 33 & 44 \\ 55 & 66 \end{pmatrix}.$$

Equipped with these definitions, we can construct any _linear combination_ of matrices:

$$r \cdot A + s \cdot B.$$

4/25/19

② _Vectors in $\mathbb{R}^n$_

Recall from Calculus that by convention, the starting point of any vector by default is at the origin.

Then a vector is fully defined by specifying its end point coordinates, x_1 & x_2.

So: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

In the same way, one defines <u>any ordered list</u> of numbers to be a vector:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leftarrow \text{"}n\text{-dimensional vector"}$$

A collection of all such vectors is $\mathbb{R}^n$.

"real" $\nearrow$

Formal writing:

$$\mathbb{R}^n = \left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ x_1, x_2, \dots x_n \ \text{are all real numbers} \right\}.$$

See also the Clarification posted after Sec. 1.5 notes.

MUST READ ON YOUR OWN

Ex. 2, 3 in textbook about the <u>vector form</u> of the solution of a l.s. !

③ <u>Matrix-vector multiplication</u>

It is defined to provide a convenient tool for writing down a l.s. in compact form.

Want to mimic a single equation:

$$a \cdot x = b$$

Coefficient $\nearrow$ $\quad$ $\uparrow$ unknown $\quad$ $\nwarrow$ constant

Now consider a l.s.

$$a_{11} x_1 + a_{12} x_2 = b_1$$
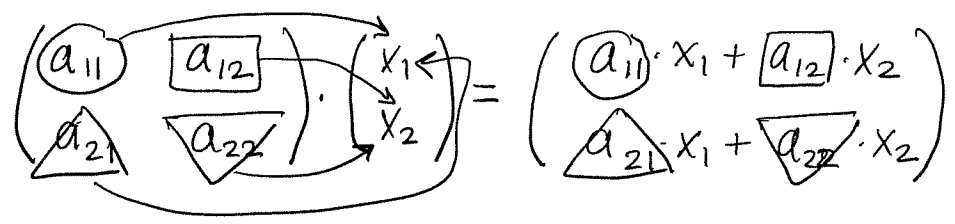
$$a_{21} x_1 + a_{22} x_2 = b_2$$

It has:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

coefficient matrix   unknown vector   const. vector

So, mimicking the single-eq. case, we want to write the l.s. as:

$$A \cdot \underline{x} = \underline{b}$$

This will be so if we define matrix-vector multiplication as:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{pmatrix}$$

In "Sigma-notation":

$$a_{11} x_1 + a_{12} x_2 = \sum_{j=1}^{2} a_{1j} x_j, \quad \text{or more generally:}$$

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = \sum_{j=1}^{n} a_{1j} x_j.$$

So:

$$A \cdot \underline{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}.$$

Then the general l.s.

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$$

can be written as $\underline{\underline{MATRIX\ FORM}}$ of a linear system

$$\boxed{A \cdot \underline{x} = \underline{b}}, \quad \text{as desired.}$$

See Ex. 6 in ~~book~~ for numbers,

See another Clarification posted after Sec.1.5 notes.

④ <u>Matrix – matrix multiplication</u>

Let A be $m \times \underset{\text{same}}{n}$, B be $n \times s$.

Note that:

$$B \equiv \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} = [\, \underline{b}_1, \underline{b}_2, \cdots \underline{b}_s \,],$$

where

$$\underline{b}_k = \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{pmatrix} \leftarrow K \underline{th}\ \underline{column}\ \text{of } B.$$

Then $\boxed{AB \equiv A [\, \underline{b}_1, \underline{b}_2, \cdots \underline{b}_s \,] = [\, A\underline{b}_1, A\underline{b}_2, \cdots A\underline{b}_s \,]}$

i.e., we simply multiply <u>each</u> <u>column of</u> B, $\underline{b}_k$, by A.

Thus, using our knowledge of matrix-vector multiplication, we can present matrix-matrix multiplication as:

$$(A \cdot B)_{\ell k} = \begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ & & \\ \boxed{a_{\ell 1} \cdots\cdots a_{\ell n}} \\ & & \\ a_{m1} & \cdots\cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & \boxed{b_{1k}} & \cdots & b_{1s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & & b_{nk} & & b_{ns} \end{pmatrix} =$$

(ℓ, k)-th entry

$$= \sum_{j=1}^{n} a_{\ell j}\, b_{jk}$$

Mnemonically: "multiply l-th <u>row</u> of the 1st matrix by the k-th <u>column</u> of the 2nd."

<u>Ex. 4</u>   Given matrices

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{pmatrix},$$

compute:

AB, BA, AC, CA, CD, DC.

<u>Sol'n:</u>   (a) $AB = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1\cdot(-3) + 2\cdot 1 & 1\cdot 2 + 2\cdot(-2) \\ 4\cdot(-3) + 3\cdot 1 & 4\cdot 2 + 3\cdot(-2) \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -9 & 2 \end{pmatrix}$$

(b) $BA = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} =$

$$= \begin{pmatrix} -3\cdot 1 + 2\cdot 4 & -3\cdot 2 + 2\cdot 3 \\ 1\cdot 1 + (-2)\cdot 4 & 1\cdot 2 + (-2)\cdot 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -7 & -4 \end{pmatrix}$$
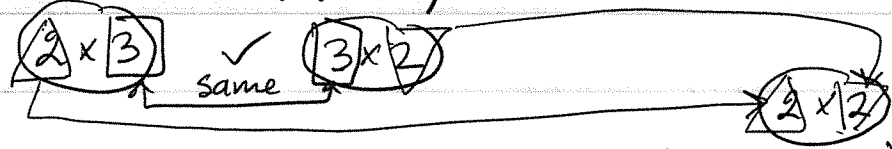
(c) $AC = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & -5 \end{pmatrix}$

$\underbrace{2\times 2}_{} \quad \underbrace{2 \times 3}_{}$

same

(d) $CA = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \ldots$ CANNOT MULTIPLY!
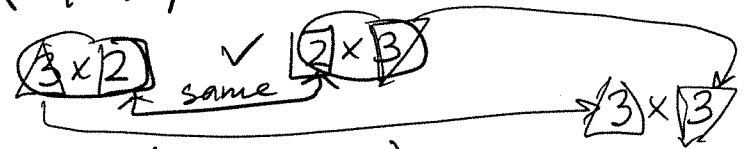
$2 \times 3 \quad \neq \quad 2 \times 2$ dimensions do not match

(e) $CD = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{pmatrix} = $

2×3 ✓ same 3×2 → 2×2

$$= \left(\begin{array}{c|c} 1\cdot 3 + 0\cdot(-1) + (-2)\cdot 1 & 1\cdot 1 + 0\cdot(-2) + (-2)\cdot 1 \\ \hline 0\cdot 3 + 1\cdot(-1) + 1\cdot 1 & 0\cdot 1 + 1\cdot(-2) + 1\cdot 1 \end{array} \right) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

(f) $DC = \begin{pmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} = $

3×2 ✓ same 2×3 → 3×3

$$\left(\begin{array}{c|c|c} 3\cdot 1 + 1\cdot 0 & 3\cdot 0 + 1\cdot 1 & 3\cdot(-2) + 1\cdot 1 \\ \hline -1\cdot 1 + (-2)\cdot 0 & -1\cdot 0 + (-2)\cdot 1 & (-1)(-2) + (-2)\cdot 1 \\ \hline 1\cdot 1 + 1\cdot 0 & 1\cdot 0 + 1\cdot 1 & 1\cdot(-2) + 1\cdot 1 \end{array} \right) = \begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

<u>Observation</u>: In general, for matrices;

$$AB \neq BA.$$

- AB may exist (c), but BA does not (d).
- Both AB and BA exist, but have different dimensions (e, f).
- AB and BA exist and have the same dimensions, but their entries are different (a, b).

⑤ <u>Alternative formulation of matrix multiplication</u>

Let's look at a l.s. in matrix form (see topic ③):

$$\underset{A}{\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}} \underset{\underline{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{\underline{b}}{\begin{pmatrix} 7 \\ 8 \end{pmatrix}}$$

$\underline{A_1} \quad \underline{A_2} \quad \underline{A_3} \quad \leftarrow$ columns of A

$$1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 = 7$$
$$4 \cdot x_1 + 5 \cdot x_2 + 6 \cdot x_3 = 8 \quad, \quad \text{or}$$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \cdot x_2 + \begin{pmatrix} 3 \\ 6 \end{pmatrix} x_3 = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

From this we derive <u>2 important conclusions</u>:

① $$A \cdot \underline{x} \equiv [\underline{A_1}, \underline{A_2}, \underline{A_3}] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \underline{A_1} \cdot x_1 + \underline{A_2} \cdot x_2 + \underline{A_3} x_3$$

In general, if A is $m \times n$ and $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then

$$\boxed{A \cdot \underline{x} = x_1 \underline{A_1} + x_2 \underline{A_2} + \dots + x_n \underline{A_n}} \quad \textbf{!}$$

Key Formula

MUST MEMORIZE!

$[\underline{A_1}, \underline{A_2}, \dots \underline{A_n}] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

See Ex. 7 in textbook, Thm. 5, and the example after it for further illustration of the Key Formula.

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2 (Corollary of the Key Formula)

If $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a solution of a l.s.

$A \cdot \underline{x} = \underline{b}$, then:

$$\boxed{x_1 \underline{A_1} + x_2 \cdot \underline{A_2} + \ldots + x_n \underline{A_n} = \underline{b}}$$

The vector equation equivalent to l.s. $A\underline{x} = \underline{b}$

In words: $\underline{b}$ is a linear combination of the columns of A.

For example, in the above Ex. 5,

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with $x_1 = -13/3$, $x_2 = 8/3$, $x_3 = 2$

is a solution of the l.s. (one of ∞ many, as per Corollary of Thm. 3 (p. 3-2 of Notes for Sec. 1.3)), then

$$-\frac{13}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{8}{3} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$x_1 \cdot \underline{A_1} \qquad + \quad x_2 \cdot \underline{A_2} \quad + \quad x_3 \cdot \underline{A_3} \quad = \quad \underline{b}$$

⑥ Solving a "matrix equation"

Ex. 6   Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Find B s.t. $AB = C$.

Sol'n: 1) Analyze dimensions.

$$\underset{2\times 2}{A} \cdot \underset{m \times \textcircled{n}}{B} = \underset{2 \times \textcircled{2}}{C}$$

$\Rightarrow n = 2$

$\Rightarrow m = 2$

Thus B must be 2×2, and so we can write $B = [\underline{B_1}, \underline{B_2}]$ 2×1 columns of B.

2) Similarly, $C = [\underline{C_1}, \underline{C_2}] \equiv \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

Then $AB = C$ can be written as

$A[\underline{B_1}, \underline{B_2}] = [\underline{C_1}, \underline{C_2}] \Rightarrow [A\underline{B_1}, A\underline{B_2}] = [\underline{C_1}, \underline{C_2}]$

$\Rightarrow A\underline{B_1} = \underline{C_1}$ and $A\underline{B_2} = \underline{C_2}$.

Thus, we need to solve two l.s.

3) $A\underline{B_1} = \underline{C_1}$
$\hookrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & | & 1 \\ 1 & 2 & | & 0 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -1 \end{pmatrix} \Rightarrow \underline{B_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

4) $A\underline{B_2} = \underline{C_2}$:

$\begin{pmatrix} 1 & 1 & | & 0 \\ 1 & 2 & | & 1 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 1 \end{pmatrix} \Rightarrow \underline{B_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow B = [\underline{B_1}, \underline{B_2}] = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$

See also Thm. 6 in book about the same method.